Connected Metrizable Topological Manifolds Are Second Countable

(*Second Version, with complete proofs*)

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Since second countable topological manifolds are paracompact, Smirnov's Theorem implies that a a second countable topological manifold is metrizable. More generally, one can ask which topological manifolds are metrizable. The first observation is that this reduces quickly to the connected case because an arbitrary topological manifold is homeomorphic to the disjoint union of its components and a disjoint union of topological spaces is metrizable if and only if each of the summands is metrizable (*Proof:* The $\Rightarrow$ implication is obvious; to prove the other direction, note that the one can find metrics on the pieces with diameter ≤ 1 that define the same topologies as the original metrics, and then one can define a metric on the disjoint union by taking the bounded metric for points in the same piece and defining the distance between two points in different pieces to be 2). Therefore the question reduces to determining which connected topological manifolds are metrizable. Here is the main result.

Theorem I. *If X is a connected topological n-manifold that is metrizable, then X is second countable.*

Corollary. *If X is a topological n-manifold, then X is metrizable if and only if each component of X is second countable.*

This follows immediately because the components of a topological n-manifold are also topological n-manifolds. Note that a disoint union of uncountably many copies of some (nonempty) topological n-manifold is metrizable but not second countable.

The Crucial Steps

A theorem due to A. H. Stone's Theorem shows that every metric space is paracompact. (see the references listed below). Theorem I turns out to be a simple consequence of A. H. Stone's Theorem and the following result from point set topology:

Theorem II. *Let X be a space that is paracompact $\mathbf{T}_2$, locally compact and connected. Then there is a countable family of compact subsets $K_n \subset X$ such that $X = \cup_n K_n$.*

The proof of Theorem II requires the following auxiliary result.

Lemma. *Let X be a topological space, let K be a compact subset of X, and let $\{U_\alpha\}$ be a locally finite open covering of X. Then there are only finitely many open sets U_β in the open covering such that $K \cap U_\beta = \emptyset$.*

Proof of the Lemma. For each $x \in K$ there is an open neighborhood V_x whose intersection with all but finitely many of the sets U_α is empty. By compactness K is contained in a finite union of the form

$$V_{x_1} \cup \cdots V_{k_m}$$

and the intersection of this finite union with U_α is empty for all but finitely many α. Therefore the intersection of K with U_α is also empty for all but finitely many α.

Proof of Theorem II. Let $\{U_\alpha\}$ be an open covering of X by subsets whose closures are compact. Such a covering exists because X is locally compact. Since every open covering has a locally finite refinement, we may as well assume that $\{U_\alpha\}$ itself is locally finite (note that the condition about compact closures is true for refinements of an open covering if it is true for the covering itself).

Choose W_0 to be an arbitrary nonempty set U_β from the open covering. Define a sequence of subspaces $\{W_n\}$ recursively by

$$W_n = \cup\{U_\alpha | U_\alpha \cap W_{n-1} \neq \emptyset\}.$$

By construction this is an increasing sequence of open subsets. We claim that $X = \cup_k W_n$. Since the right hand side is nonempty and open, it suffices to show that $\cup_k W_n$ is closed. Suppose that x lies in the closure of $\cup_k W_n$. Then $x \in U_\alpha$ for some α, and since the closure of a set is the union of that set and its limit points it follows that

$$U_\alpha \cap (\cup_k W_n) \neq \emptyset.$$

The latter in turn implies that $U_\alpha \cap W_{n_0} \neq \emptyset$ for some n_0. But this implies that $x \in W_{n_0+1}$. Therefore all points in the closure $\overline{\cup_k W_n}$ in fact lie in $\cup_n W_n$, and hence the latter is closed.

We shall now show that the sets $\overline{W_n}$ is compact by induction on n; if $k = 0$ this holds because $\overline{U_\beta}$ is compact. If $\overline{W_n}$ compact, then by the lemma there are only finitely many U_α such that $U_\alpha \cap \overline{W_n} \neq \emptyset$; call these $U_{\alpha_1}, \cdots, U_{\alpha_p}$. It then follows that

$$\overline{W_{n+1}} = \overline{U_{\alpha_1}} \cup \cdots \overline{\cup U_{\alpha_p}}.$$

Since each of the closures on the right hand side is compact, it follows that the left hand side is a finite union of compact subsets and therefore is compact. Therefore if we set $K_n = \overline{W_n}$ then we know that K_n is compact, $K_n \subset K_{n+1}$ for all n, and $X = \cup_n K_n$.

Completion of Proof of Theorem I

As noted above, X is paracompact if X is metrizable. Therefore by Theorem II we know that $X = \cup K_n$ where each K_n is compact. Furthermore, since X is metrizable each K_n is also metrizable. The latter implies that each K_n has a countable dense subset D_n, and therefore the countable subset $\cup D_n$ is dense in X. Since X is metric, this means that it is also second countable.

References

[D] J. Dugundji, *Topology.* Allyn and Bacon, Boston, 1966.

[K] J. L. Kelley, *General Topology.* Van Nostrand, New York, 1955.