Mathematics 205C
Spring 2003

Study sheet for Midterm Examination

1. Show that the set S of all points (x, y, z) in $\mathbb{R}^3$ satisfying $x^2 + y^2 = z^2$ is ***not*** a topological manifold. [*Hint:* The origin 0 lies in S and $S - \{0\}$ has two components; show that for every open neighborhood U of 0 in S the deleted neighborhood $U - \{0\}$ has at least two components. Why does this imply that S cannot be a topological n-manifold for $n \geq 2$? On the other hand, the set of all points in $S - \{0\}$ with positive z-coordinate is homeomorphic to $\mathbb{R}^2 - \{0\}$ by the map forgetting the last coordinate. Since the latter is a topological 2-manifold, it follows that $S - \{0\}$, and hence S itself, cannot be a topological 1-manifold.]

A simpler question along the same lines is to show that the set of points in the plane satisfying $x^2 = y^2$ is not a topological manifold. As in class, this can be done as follows: In a topological manifold of dimension greater than 1, for every point p there are arbitrarily small open neighborhoods that are connected and remain connected if the point p is removed, and for 1-manifolds there are arbitrarily small connected neighborhoods that split into two components when p is removed. On the other hand, for the set described above, if U is an arbitrary neighborhood of the origin, then $U - \{0\}$ always has at least four components.

2. Let U and V be open subsets of Euclidean spaces, and let $f : U \to V$ and $g : V \to U$ be smooth maps such that $gf = \text{Identity}(U)$. Prove that f is an immersion, f is one-to-one, and f is a closed mapping. [*Hint:* The relation $gf = \text{Identity}$ implies that f is one-to-one and f is an immersion; to see that it is closed, show that if F is closed in U then $f(F)$ is the set of all points in $g^{-1}(F)$ such that $fg(x) = x$. Why is this intersection closed in V?]

3. Outline the construction of a smooth real valued function f on the real line such that $f(x)$ is zero when x is nonpositive and 1 if x is greater than or equal to 1.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth map, let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth map, and let X be the smooth vector field defined by $X(u) = (u, G(u))$. If $G(u)$ is perpendicular to the gradient of $\nabla f(u)$ for all u, show that the integral curves $\gamma(t)$ of X satisfy $f(\gamma(t)) = \text{constant}$. Using this, find a vector field whose integral curves satisfy the equation

$$x^3 + y^3 = C.$$

[*Hint:* Find the gradient of the left hand side and use the fact that $(B, -A)$ is perpendicular to (A, B).]

5. Define a smooth atlas $\mathscr{A}$, and state the compatibility condition that characterizes the unique maximal atlas $\mathscr{M}$ containing $\mathscr{A}$. Let U be an open subset of Euclidean space and let $\mathscr{A}$ be the atlas consisting only of the identity map of U. Given a diffeomorphism $g: V \to U$ where V is open in the Euclidean space containing U, show that the chart (V, g) belongs to $\mathscr{M}$.

6. Let M and N be smooth manifolds (the maximal atlases will be suppressed for notational simplicity). Using the characterization of product manifolds in terms of morphisms, prove the following:

(*i*) If $f: M \to M'$ and $g: N \to N'$ are smooth then so is the product $f \times g$. If f and g are diffeomorphisms then so is $f \times g$. What is its inverse in this case? [*Hint:* Look at the projections onto the first and second coordinates.]

(*ii*) The twist map $T: M \times M \to M \times M$ given by $T(x, y) = (y, x)$ is a diffeomorphism that is equal to its own inverse.

(*iii*) If y is an arbitrary point of N show that the slice injection $s_y: M \to M \times N$ defined by $s_y(x) = (x, y)$ is smooth. [*Hint:* Constant maps are always smooth.]

7. Let $f: \mathbf{R}^2 \to \mathbf{R}^2$ be defined by $f(x, y) = (x + \sinh y, y)$. Prove that f is a diffeomorphism. [*Hint:* It is necessary to show that the map is continuous, one-to-one and onto, has continuous partials of all orders, and has a nonvanishing Jacobian determinant at each point.]