Orientability of Manifolds

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Abstract

This brief discussion of orientability for differentiable manifolds follows a manuscript of lecture notes by Prof. Peter Petersen, UCLA. The presentation uses the point of view of covering spaces. In this talk we will explore this viewpoint and establish its equivalence with the viewpoint we discussed last semester.

Let us first review some linear algebra. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two ordered bases of a finite dimensional vector space V.

Definition: $\mathcal{B}_1$ and $\mathcal{B}_2$ represent the same **orientation** for V if the transition matrix M_{21} from $\mathcal{B}_1$ to $\mathcal{B}_2$ has positive determinant.

This relation, call it $\sim$, on the set of all ordered bases of V is clearly reflexive since $M_{11} = I$, and symmetric since $\det M_{12} = \det M_{21}^{-1} = (\det M_{21})^{-1} > 0$. If $\mathcal{B}_3$ is a third ordered basis for V and $\mathcal{B}_1 \sim \mathcal{B}_2$ and $\mathcal{B}_2 \sim \mathcal{B}_3$ then $\mathcal{B}_1 \sim \mathcal{B}_3$ since

$$\det M_{31} = \det(M_{32}M_{21}) = \det M_{32} \cdot \det M_{21}.$$

So $\sim$ gives an equivalence relation; furthermore, it has exactly two equivalence classes.

Definition: A choice of such an equivalence class is called an **orientation** for the vector space V.

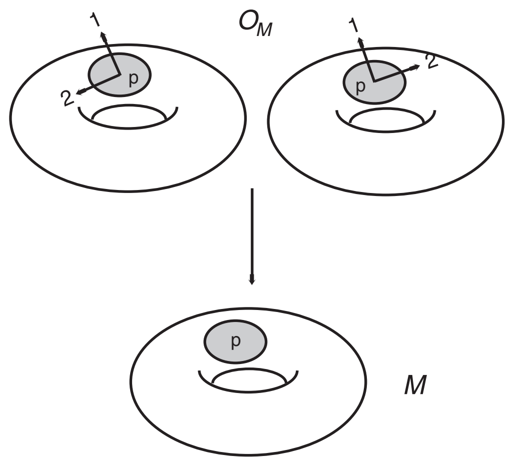
If M is a smooth manifold of dimension n then for each $p \in M$ the tangent space T_pM is a vector space of dimension n, and hence has two choices of orientation. We would like to use this scenario to construct a two-sheeted covering space O_M called the **orientation covering** of M. If $(x^1, ..., x^n)$ are coordinates on a connected neighborhood $U \subset M$, then the set $\{\partial_1|_p, ..., \partial_n|_p\}$ form a basis for the tangent space at each $p \in M$, where

$$\partial_i|_p = \left.\frac{\partial}{\partial x^i}\right|_p .$$

Since the choice of basis depends smoothly on p, we say that the n-tuple $(\partial_1, ..., \partial_n)$ gives a **framing** for the tangent bundle over U. By an orientation over U we will mean a choice of such a framing. Since U is connected we have two choices for orientation over U , i.e. the classes determined by the framing $(\partial_1, ..., \partial_n)$ and $(-\partial_1, \partial_2, ..., \partial_n)$. This analysis yields a two-fold covering map $O_M \to M$ where the preimage of each $p \in M$ consists of the two orientations over p.

Definition: A connected manifold M is **orientable** if and only if its orientation covering O_M is disconnected.

Example: To clear up the subtleties of the above discussion let us consider the 2-Torus, which we know to be an orientable manifold. Then the orientation covering takes the form as shown in the picture below.



An advantage of this covering space point of view is that we immediately have the following result.

Proposition: If M is a smooth connected manifold with $\pi_1(M) = 0$ then M is orientable.

Proof: Each covering space $\tilde{M} \to M$ is trivial since if $p \in M$ then $\pi_1(\tilde{M}, \tilde{p}) \subset \pi_1(M, p) = 0$. In particular the orientation covering must then consist of two simply-connected components, each diffeomorphic to M.

Thus S^n is orientable for $n > 1$. Another observation we may make is that the orientation covering is an orientable manifold since it is locally the same as M and an orientation on the tangent space has been chosen for us.

Theorem: Let M be a smooth connected manifold of dimension n, then the following are equivalent:

1. M is orientable.
2. Orientation is preserved moving along loops in M.
3. M admits an atlas where the Jacobians of all the transition functions are positive.
4. There exists a nowhere vanishing n-form on M.

Proof: $1 \Leftrightarrow 2$: The unique path lifting property for the covering $O_M \to M$ implies that orientation is preserved along loops if and only if O_M is disconnected.

$1 \Rightarrow 3$: Choose an orientation for M, and let $\{(U_\alpha, \phi_\alpha) : \alpha \in A\}$ be an atlas for M where U_α is connected for each $\alpha \in A$. From the discussion above we see that ϕ_α either corresponds to the chosen orientation, or otherwise differs by a minus sign in the first component. If the latter is true we merely create a new chart by changing the sign of the first component of ϕ_α. Using this procedure we get an atlas where each chart corresponds to the chosen orientation. Then if ϕ_α and ϕ_β are coordinate charts in this new atlas, the transition map $\phi_\alpha \circ \phi_\beta^{-1}$ has positive Jacobian since it preserves the canonical orientation of $\mathbb{R}^n$.

$3 \Rightarrow 4$: Let $\{(U_\alpha, \phi_\alpha) : \alpha \in A\}$ be an atlas for M for which the Jacobian of each transition map is positive, and choose a locally finite partition of unity $\{\psi_\alpha\}$ subordinate to the cover $\{U_\alpha\}$. On each U_α we have the nowhere vanishing form $\omega_\alpha = dx_\alpha^1 \wedge ... \wedge dx_\alpha^n$. Notice that on an overlap $U_\alpha \cap U_\beta$, we may write the *alpha*-coordinates $x_\alpha^1 = x_\alpha^1(x_\beta^1, ..., x_\beta^n)$, ..., $x_\alpha^n = x_\alpha^n(x_\beta^1, ..., x_\beta^n)$ in terms of the β-coordinates via the transition map $\phi_\alpha \circ \phi_\beta^{-1}$. By the chain rule $\frac{\partial}{\partial x_\beta^i} = \sum_j \frac{\partial x_\alpha^j}{\partial x_\beta^i} \frac{\partial}{\partial x_\alpha^j}$, and thus we have

$$dx_\alpha^1 \wedge ... \wedge dx_\alpha^n \left(\frac{\partial}{\partial x_\beta^1}, ..., \frac{\partial}{\partial x_\beta^n} \right)$$
$$= \det \begin{bmatrix} dx_\alpha^1\left(\frac{\partial}{\partial x_\beta^1}\right) & \dots & dx_\alpha^1\left(\frac{\partial}{\partial x_\beta^n}\right) \\ \vdots & \ddots & \vdots \\ dx_\alpha^n\left(\frac{\partial}{\partial x_\beta^1}\right) & \dots & dx_\alpha^n\left(\frac{\partial}{\partial x_\beta^n}\right) \end{bmatrix}$$

$$
\begin{aligned}
&= \det \begin{bmatrix} dx^1_\alpha\left(\sum_j \frac{\partial x^j_\alpha}{\partial x^1_\beta}\frac{\partial}{\partial x^j_\alpha}\right) & \dots & dx^1_\alpha\left(\sum_j \frac{\partial x^j_\alpha}{\partial x^n_\beta}\frac{\partial}{\partial x^j_\alpha}\right) \\ \vdots & \ddots & \vdots \\ dx^n_\alpha\left(\sum_j \frac{\partial x^j_\alpha}{\partial x^1_\beta}\frac{\partial}{\partial x^j_\alpha}\right) & \dots & dx^n_\alpha\left(\sum_j \frac{\partial x^j_\alpha}{\partial x^n_\beta}\frac{\partial}{\partial x^j_\alpha}\right) \end{bmatrix} \\
&= \det \begin{bmatrix} \frac{\partial x^1_\alpha}{\partial x^1_\beta} & \dots & \frac{\partial x^1_\alpha}{\partial x^n_\beta} \\ \vdots & \ddots & \vdots \\ \frac{\partial x^n_\alpha}{\partial x^1_\beta} & \dots & \frac{\partial x^n_\alpha}{\partial x^n_\beta} \end{bmatrix} \\
&= \det D(\phi_\alpha \circ \phi_\beta^{-1}) \\
&> 0.
\end{aligned}
$$

Hence the globally defined form $\sum_\alpha \psi_\alpha \omega_\alpha$ is always non-negative when evaluated on the frame $(\partial/\partial x^1_\beta, ..., \partial/\partial x^n_\beta)$. Further, at least one term must be non-zero since $\{\psi_\alpha\}$ is a partition of unity.

$4 \Rightarrow 1$: Let ω be a non-vanishing n-form on M. Define the set O_+ to be the set of all points $p \in M$ such that $\omega_p > 0$ when evaluated on a basis, and conversely define O_- according to when $\omega_p < 0$ when evaluated on a basis. This yields two disjoint open sets in O_M which cover all of M, hence O_M is disconnected and M is orientable.

The above result gives good conditions for establishing that a manifold we expect to be orientable is orientable. However, establishing that a manifold is not orientable can be tricky. The following is one of the greatest advantages to the covering space point of view for this topic.

In particular, notice that if we suspect that a manifold M is non-orientable, then our definition of orientability of a manifold implies that there exists a connected 2-fold covering $\pi\colon \hat{M} \to M$, where $\hat{M}$ is oriented and the two orientations at points over $p \in M$ are mapped different orientations in M via $D\pi$. Recall that for such a two-fold cover, there always exists a deck transformation $\delta\colon \hat{M} \to \hat{M}$ with the properties that $\delta(x) \neq x$, $\delta \circ \delta = \mathrm{id}_{\hat{M}}$, and $\pi \circ \delta = \pi$. This gives rise to the Double Covering Lemma:

Lemma: (Double Covering) Let $\pi\colon \hat{M} \to M$ be a connected 2-fold covering of a connected n-manifold M, where $\hat{M}$ is an oriented manifold, then M is orientable if and only if the deck transformation $\delta\colon \hat{M} \to \hat{M}$ is orientation preserving.

<u>Proof:</u> Suppose that δ preserves the orientation on $\hat{M}$, and let $p \in M$, and denote $\{p_1, p_2\} = \pi^{-1}\{p\}$. Then given a choice of orientation $e_1, ..., e_n \in T_{p_1}\hat{M}$ we can declare that $D\pi(e_1), ..., D\pi(e_n) \in T_pM$ to be an orientation at p. This declaration is consistent since $\pi \circ \delta = \pi$ implies that $(D\delta)(e_1), ..., (D\delta)(e_n) \in T_{p_2}\hat{M}$ which represents the orientation on $\hat{M}$ is mapped to $D\pi(e_1), ..., D\pi(e_n)$.

Conversely, let M be orientable and choose an orientation for M. Since $\hat{M}$ and M are connected and $D\pi$ is non-singular for each $p \in M$, we have that $D\pi$ is either globally orientation preserving or globally orientation reversing. Without loss of generality we may assume that the orientation is preserved. Therefore δ must preserve orientation since $\pi = \pi \circ \delta$ implies that $0 < \det(D\pi)(p_1) = \det(D(\pi \circ \delta))(p_1) = \det(D\pi)(\delta(p_1)) \cdot \det D\delta(p_1) = \det(D\pi)(p_2) \cdot \det(D\delta)(p_1)$. Hence δ must preserve orientation on $\hat{M}$.

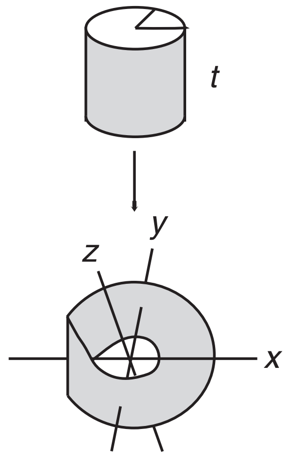
To get an idea of how this machinery works let's check some examples with which we are familiar. We've already seen that S^n is orientable for each $n > 1$ using the covering space defintion. Another way of showing orientability of the spheres is to construct a non-vanishing n-form. This can be done by considering the canonical embedding of S^n as the unit sphere $\mathbb{R}^{n+1}$, and contracting the volume form $dx^1 \wedge ... \wedge dx^{n+1}$ in the direction of the of the radial vector field $X = \sum_i x^i \partial_i$. i.e., $\omega = i_X(dx^1 \wedge ... \wedge dx^{n+1})$. Since $X|_{S^n}$ is a vector field normal to the sphere, if $\{v_2, ..., v_n\}$ form a basis for the tangent space at $p \in S^n$, then $\{X_p, v_2, ..., v_n\}$ form a basis for $\mathbb{R}^{n+1}$ and hence

$$\begin{aligned} \omega(v_2, ..., v_n) \qquad &= i_X(dx^1 \wedge ... \wedge dx^{n+1})(v_1, ..., v_n) \\ &= dx^1 \wedge ... \wedge dx^{n+1}(X, v_2, ..., v_n) \\ &\neq 0 \end{aligned}$$

Example: Recall that S^n is a natural double covering of $\mathbb{R}\mathbf{P}^n$ with the antipodal map $a\colon S^n \to S^n$ as deck transformation. The antipodal map preserves the radial vector field X, i.e. $X|_p$ and $X|_{a(p)}$ are both normal to S^n at p. So A preserves the orientation of S^n if and only if the determinant of its extension to $\mathbb{R}^{n+1}$ is positive. This only happens if $n+1$ is even. Thus we can conclude that $\mathbb{R}\mathbf{P}^n$ is orientable if and only if n is odd.

Example: Consider the following double covering of the open Möbius Strip M by the open cylinder $\mathcal{C} = \{(r, \theta, t) \in \mathbb{R}^3 : r = 1, \theta \in [0, 2\pi), t \in (-1, 1)\}$ expressed in cylindrical coordinates on $\mathbb{R}^3$, given by the covering map $\phi(\theta, t) = (x, y, z)$ where

$$\begin{aligned} x(\theta, t) \quad &= 2\cos 2\theta + t \cos\theta \cos 2\theta \\ y(\theta, t) \quad &= 2\sin 2\theta + t \cos\theta \sin 2\theta \\ z(\theta, t) \quad &= t \sin\theta \end{aligned}$$



The deck transformation I which we are looking for is given by the antipodal map $a\colon \mathcal{C} \to \mathcal{C}$ via $a(\theta, t) = (\theta+\pi, -t)$. To show that this is a deck transformation we need to check that $\phi(a(\theta,t)) = \phi(\theta,t)$. Notice that

$$\begin{aligned} x(a(\theta,t)) \quad &= x(\theta+\pi, -t) \quad &&= 2\cos 2(\theta+\pi) - t\cos(\theta+\pi)\cos 2(\theta+\pi) \\ &&&= 2\cos 2\theta - t(-\cos\theta)\cos 2\theta \\ &&&= x(\theta,t). \end{aligned}$$

A similar computation shows that the result holds for y and z as well. If we give the cylinder the orientation induced from the canonical orientation on $\mathbb{R}^3$, then by the Double Covering Lemma, M will be non-orientable if a is orientation reversing. A simple computation shows that

$$\det Da(\theta,t) = \det \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = -1 < 0$$

hence M is non-orientable.