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Proper Maps and Universally Closed Maps

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Continuous functions defined on compact spaces generally have special properties. For example, if X is a compact space and Y is a Hausdorff space, then f is a closed mapping. There is a useful class of continuous mappings called *proper*, *perfect*, or *compact* maps that satisfy many properties of continuous maps with compact domains. In these notes we shall study a few basic properties of these maps. Further information about proper maps can be found in Bourbaki [B_2], Section I.10), Dugundji [D] (Sections XI.5–6, pages 235–240), and Kasriel [K] (Sections 95 and 105, pages 214–217 and 243–247). There is a slight difference between the notions of perfect and compact mappings in [B_2], [D], and [K] and the notion of proper map considered here. Specifically, the definitions in [D] and [K] include a requirement that the maps in question be surjective. The definition in [B_2] is entirely different and corresponds to the notion of *universally closed* map discussed later in these notes; the equivalence of this definition with ours for a reasonable class of spaces is proved both in [B_2] and in these notes (see the section *Universally closed maps*).

Proper maps

If $f : X \to Y$ is a continuous map of topological spaces, then f is said to be *proper* if for each compact subset $K \subset Y$ the inverse image $f^{-1}(K)$ is also compact. Of course, if X and Y are Hausdorff then the continuity assumption is redundant.

Examples. Suppose that A is a finite subset of $\mathbb{R}^n$ and $f : \mathbb{R}^n - A \to \mathbb{R}^m$ is a continuous function such that for all $a \in A$ we have

$$\lim_{x \to a} |f(x)| = \lim_{x \to \infty} |f(x)| = \infty.$$

Then f is proper. Proof: If $K \subset \mathbb{R}^m$ is compact, then K is contained in some large disk D. The limit condition at ∞ implies that $f^{-1}(D)$ is contained in some large disk $D' \subset R^n$; furthermore the limit conditions at all the points $a_i \in A$ imply that $f^{-1}(D)$ is also contained in the complement of $E = D' - \cup N_i$, where N_i is a small open disk neighborhood with center a_i. But E is compact and $f^{-1}(K)$ is a closed subset of $\mathbb{R}^n$ that is contained in E; it follows that $f^{-1}(K)$ is compact.■

Important special case: Suppose $n = m = 2$ and $f(z) = p(z)/q(z)$, where p and q are complex polynomials with $\deg p > \deg q$. Then f satisfies the limit condition and therefore is proper.

Theorem 1. *Suppose that X and Y are Hausdorff spaces with Y either locally compact or metrizable, and let $f : X \to Y$ be continuous. Then f is proper if and only if f is closed and for each $y \in Y$ the set $f^{-1}(\{y\})$ is compact.*

Proof. Suppose that f is closed and inverse images of one point sets are compact; we claim that inverse images of arbitrary compact subsets are compact.

Let K be a compact subset of Y, and let $\mathcal{F} = \{F_\alpha\}$ be a collection of closed subsets of $f^{-1}(K)$ with the finite intersection property. Without loss of generality, we may assume that

$\mathcal{F}$ is closed under finite intersections, for if $\mathcal{F}'$ is the family of all finite intersections of sets in $\mathcal{F}$ then $\mathcal{F}'$ is closed under finite intersections and the intersection of all the sets in $\mathcal{F}$ equals the intersection of all the sets in $\mathcal{F}'$. Since f is closed, the subsets $f(F_\alpha)$ are closed in Y; since $F_\alpha \subset f^{-1}(K)$ is true by assumption, it follows that $f(F_\alpha) \subset K$. Combining these observations, we see that $f(F_\alpha)$ is a compact subset of K. But the family of closed sets $f(F_\alpha)$ also has the finite intersection property because

$$\varnothing \neq f(F_{\alpha_1} \cap \ldots \cap f(F_{\alpha_k}) \subset f(F_{\alpha_1}) \cap \ldots \cap f(F_{\alpha_k}).$$

Therefore $\varnothing \neq \cap f(F_\alpha)$ by compactness of K. Let y be a point in the intersection. Since the family $\mathcal{F}$ is closed under finite intersections, for all $\alpha_1, \ldots, \alpha_n$ we have $f^{-1}(\{y\} \cap F_{\alpha_1} \cap \ldots \cap F_{\alpha_n} \neq \varnothing$. Therefore the family $\{ f^{-1}(\{y\}) \cap F_\alpha\}$ has the finite intersection property. But $f^{-1}(\{y\})$ is compact, and therefore $\varnothing \neq f^{-1}(\{y\}) \cap (\cap_\alpha F_\alpha) \subset \cap_\alpha F_\alpha$. Therefore the set $f^{-1}(K)$ is compact. $\square$

Suppose now that f is proper and Y is locally compact or metrizable. If $F \subset X$ is a closed subset, then it is immediate that $f|F$ is proper. Therefore it suffices to prove that if f is proper then $f(X)$ is closed in Y (because $f(F) = f|F\,(F)$).

Assume first that Y is locally compact. Let y be a point in the closure of $f(X)$, and let K be a compact neighborhood of y. Then $f^{-1}(K)$ is a nonempty compact set, and $f|\,f^{-1}(K)$ a closed mapping. Therefore $f(\,f^{-1}(K)\,) = f(X) \cap K$ is a closed set. But by construction y is a limit point of this set, and consequently $y \in f(X) \cap K \subset f(X)$. $\square$

Assume now that Y is a metric space. Let F be closed in X, and let $\{x_n\}$ be a sequence of points in F such that $\lim f(x_n) = y$. Let C_n be the compact set $\{y, f(x_n), f(x_{n+1}), \ldots), $ so that $f^{-1}(C_n)$ is a nonempty nested sequence of compact sets. Therefore by compactness we have that $\varnothing \neq \cap_n(\,f^{-1}(C_n) \cap F\,) = (\,f^{-1}(\cap_n C_n)\,) \ \cap \ F = f^{-1}(\{y\}) \cap F$. Therefore $y \in f(F)$ must hold, and consequently $f(F)$ must be closed in Y.■

Remark. The ($\Leftarrow$) implication does not require an extra condition on Y. The ($\Rightarrow$) implication is valid more generally if Y is a k-space (see [D], Section XI.9, especially XI.9.3 on p. 248). Here is the proof that f is closed under this hypothesis: The set $f(X)$ is closed if and only if $f(X) \cap K$ is compact for all compact subsets $K \subset Y$. But $f(X) \cap K = f(\,f^{-1}(K)\,)$, and this set is compact because f proper $\Rightarrow$ $f^{-1}(K)$ is compact and $f^{-1}(K)$ compact $\Rightarrow$ $f(\,f^{-1}(K)\,)$ compact. $\square$

Theorem 2. *Suppose that $f : X \to Y$ is a proper map and $B \subset Y$. Then $f|f^{-1}(B)$ is proper. Conversely, if $\{B_\alpha\}$ is either an open covering or a finite closed covering of Y and each of the maps $f|f^{-1}(B_\alpha)$ is proper, then f is also proper.*

Proof. ($\Rightarrow$) Let $f_B = f|f^{-1}(B)$. If K is a compact subset of B then $f_B^{-1}(K) = f^{-1}(K)$; but the latter is compact since f is proper, and therefore it follows that f_B is proper.

($\Leftarrow$) Let $f_\alpha = f|f^{-1}(B_\alpha)$, and let $K \subset Y$ be compact. Suppose that $\{B_\alpha\}$ is an open covering of Y. Then by compactness K is contained in a finite union $B_{\alpha_1}, \cup \ldots \cup B_{\alpha_k}$. Let $V_{\alpha_j} = K \cap B_{\alpha_j}$; then the sets $\{V_{\alpha_j}\}$ define a finite open covering of K, and by Munkres, Theorem 4-5.1, pp. 222-223 (especially Step 1), we can find closed (hence compact) subsets $F_{\alpha_j} \subset V_{\alpha_j}$ such that $K = \cup_j F_{\alpha_j}$. Therefore it follows that $f^{-1}(K) = \cup_j f_{\alpha_j}^{-1}(F_{\alpha_j})$. But each map $f_{\alpha_j}^{-1}$ is proper, and hence it follows that each subset of the form $f_{\alpha_j}^{-1}(F_{\alpha_j})$ is compact. Since a finite union of compact sets is compact, it follows that $f^{-1}(K)$ is compact.

Now suppose that $\{B_\alpha\}$ is a finite closed covering. Let $K_\alpha = K \cap B_\alpha$; then each set K_α is compact because each set B_α is closed in Y. Since each map f_α is proper we know that each set $f_\alpha^{-1}(K_\alpha) = f^{-1}(K_\alpha)$ is compact. But $f^{-1}(K) = \cup_\alpha f_\alpha^{-1}(K_\alpha)$, and therefore $f^{-1}(K)$ is compact because a finite union of compact sets is compact.■

Remark. The theorem also holds for *locally finite* closed coverings.

Theorem 3. *Let $f_\alpha : X_\alpha \to Y_\alpha$ be continuous for each $\alpha \in J$, and let*

$$\Pi f_\alpha : \Pi X_\alpha \to \Pi Y_\alpha$$

be the associated map of product spaces. Then Πf_α is proper if and only if each f_α is proper.

Proof. ($\Rightarrow$) Let $\beta \in J$ be arbitrary, and for each $\alpha \neq \beta$ choose $x_\alpha \in X_\alpha$. Furthermore, for each $\alpha \neq \beta$ let $y_\alpha = f_\alpha(x_\alpha)$. Let $i_\beta : X_\beta \to X_\beta \times \Pi_{\alpha\neq\beta}\{x_\alpha\}$ be the identity on the β factor and constant on the other factors, and let $j_\beta : Y_\beta \to Y_\beta \times \Pi_{\alpha\neq\beta}\{y_\alpha\}$ be the identity on the β factor and constant on the other factors; then i_β and j_β are closed subspace inclusions and

$$(\Pi f_\alpha) i_\beta = j_\beta f_\beta .$$

Let $K \subset Y_\beta$ be compact. Then the formula above implies that $i_\beta(f_\beta^{-1}(K)) = (\Pi f_\alpha)^{-1}(j_\beta(K))$, and since the product map is proper it follows that $i_\beta(f_\beta^{-1}(K))$ is compact. But i_β is a homeomorphism onto a closed subset, and therefore a subset $B \subset X_\beta$ is compact if and only if $i_\beta(B)$ is compact. Therefore it follows that $f_\beta^{-1}(K)$ is compact. □

($\Leftarrow$) Suppose that K is compact in ΠY_α. Then $L_\alpha = \pi_\alpha(L)$ is a compact subset of Y_α. By Tychonoff's Theorem the product ΠL_α is compact. It is immediate that $(\Pi f_\alpha)^{-1}(K)$ is a closed subset of $\Pi f_\alpha^{-1}(L_\alpha)$. But the factors $f_\alpha^{-1}(L_\alpha)$ are all compact because the maps f_α are proper, and therefore the product is compact by Tychonoff's Theorem. The observations of the preceding two sentences combine to show that $(\Pi f_\alpha)^{-1}(K)$ is compact.■

Theorem 4. *Let $f : X \to Y$ and $g : Y \to Z$ be continuous.*

(*i*) *If f and g are proper, then gf is also proper.*

(*ii*) *If gf is proper, then f is proper.*

(*iii*) *If gf is proper and f is onto, then g is proper.*

Proof. (*i*) Let K be a compact subset of Z. Then $g^{-1}(K)$ is a compact subset of Y because g is proper; but f is also proper, and therefore $(gf)^{-1}(K) = f^{-1}g^{-1}(K)$ is also compact. □

(*ii*) Let C be a compact subset of Y. Then $g(C)$ is compact, and therefore $f^{-1}g^{-1}(g(C))$ is a compact subset of X. Since $C \subset g^{-1}(g(C))$ and $f^{-1}(C)$ is a closed subset of X, it follows that $f^{-1}(C)$ is a closed subset of the compact set $f^{-1}g^{-1}(g(C))$, and therefore $f^{-1}(C)$ is compact. □

(*iii*) Let K be a compact subset of Z. Since gf is proper the set $(gf)^{-1}(Z) = f^{-1}g^{-1}(Z)$ is compact. However, f is onto and therefore $B = f(f^{-1}(B))$ for all $B \subset Y$. It follows that $g^{-1}(K) = f(f^{-1}g^{-1}(K))$, and the set on the right hand side is compact because it is the image of the compact set $(gf)^{-1}(Z) = f^{-1}g^{-1}(Z)$ under the continuous function f.■

Universally closed maps

Proper maps arise naturally in many analytic and geometric contexts (compare [B$_1$] and [Se]). For these purposes it is convenient to have an alternate characterization of proper maps; in fact, the definition in [B$_2$] takes this characterization as the definition of proper map. Since it is difficult to find an account of this outside of [B$_2$], the details are given below.

DEFINITION. Let $\mathcal{A}$ be a family of Hausdorff spaces such that the following hold:

(*i*) If $A \in \mathcal{A}$ and B is homeomorphic to A then $B \in \mathcal{A}$.

(*ii*) $\mathcal{A}$ contains all one point spaces.

If $X, Y \in \mathcal{A}$, then a continuous map $f : X \to Y$ is said to be *universally closed* (with respect to $\mathcal{A}$) if for each $Z \in \mathcal{A}$ the map $f \times 1_Z : X \times Z \to Y \times Z$ is a closed mapping. Notice that if $\mathcal{B} \subset \mathcal{A}$ and f is $\mathcal{A}$-universally closed, then f is also $\mathcal{B}$-universally closed.

A space $X \in \mathcal{A}$ is said to be $\mathcal{A}$-*complete* if the constant map $X \to \{\text{pt.}\}$ is proper.

Lemma 5. *The composite of two $\mathcal{A}$-universally closed maps is $\mathcal{A}$-universally closed.*

The proof of this is elementary.■

Theorem 6. *Let $\mathcal{A}$ be one of the following families:*
(*i*) *All $T_{3.5}$ spaces.*
(*ii*) *All locally compact Hausdorff spaces.*
(*iii*) *All closed subsets of $\mathbb{R}^n$, where n ranges over all nonnegative integers.*
Then a space $X \in \mathcal{A}$ is $\mathcal{A}$-complete if and only if X is compact.

Proof. To prove the ($\Leftarrow$) implication, it suffices to show that if X is compact Hausdorff and Y is $T_{3.5}$ then the second coordinate projection $\pi_Y : X \times Y \to Y$ is a closed mapping.

Let F be a closed subset of $X \times Y$, and let $W = X \times Y - F$; we claim that W is open. If $y \notin \pi_Y(F)$, then for every $x \in X$ there is an open subset $U_x \times V_x \subset X \times Y$ such that $x \in U_x$, $y \in V_x$, and $U_x \times V_x \cap F = \varnothing$. The family $\{U_x\}$ forms an open covering of $X \times \{y\}$, and by compactness of X there is a finite subcovering $U_{x_1}, \ldots U_{x_k}$. Let $V = \cap V_{x_i}$. Then $\cup(U_{x_i} \times V) \cap F = \varnothing$, and consequently $V \cap \pi_Y(F) = \varnothing$, so that $y \in V$. □

The proof of the ($\Rightarrow$) implication separates into three cases depending on $\mathcal{A}$.

Case 1. Suppose that $\mathcal{A}$ is all $T_{3.5}$ spaces. If X is $T_{3.5}$ but not compact, then X is homeomorphic to a proper subset of its Stone-Čech compactification. Let $F \subset X \times \beta X$ be the graph of the embedding of X in βX. Then F is closed in $X \times \beta X$, but the projection $\pi_{\beta X}(X) \subset \beta X$ is a dense proper subset. □

Case 2. Suppose that $\mathcal{A}$ is all locally compact Hausdorff spaces. The proof in Case 1 goes through if one replaces the Stone-Čech compactification βX with the one point compactification $X^\bullet$. □

Case 3. Suppose that $\mathcal{A}$ is all closed subsets of $\mathbb{R}^n$, where n ranges over all nonnegative integers. If X is a noncompact closed subset of $\mathbb{R}^k$ then its one point compactification $X^\bullet$ is a closed subspace of $(\mathbb{R}^k)^\bullet \approx S^n \subset \mathbb{R}^{k+1}$. It follows that $X^\bullet \in \mathcal{A}$, and this implies that the argument in Case 1 can be modified to apply here, with $X^\bullet$ repacing βX.■

Theorem 7. *Let $\mathcal{A}$ be either the set of locally compact Hausdorff spaces or the set of closed subspaces of the spaces $\mathbb{R}^k$. Then a continuous map $f : X \to Y$ of spaces in $\mathcal{A}$ is $\mathcal{A}$-universally closed if and only if it is proper.*

Remark. If $\mathcal{A}$ is the class of all Hausdorff spaces, then there are proper maps $f : X \to Y$ such that $X, Y \in \mathcal{A}$ but f is not universally closed. One example is indicated in [B_2], Exc. I.10.4: Geven an uncountable family of topological spaces X_α, define a modified product topology Π' with subbase given by all products of open sets $U_\alpha \subset X_\alpha$ where $U_\alpha = X_\alpha$ for all but at most countably many α. If we take each X_α to be a two point space with the discrete topology, then it is immediate that $Y = \Pi' X_\alpha$ is T_3 and that the "identity" map j from $X = \Pi X_\alpha$ to Y is continuous. Since j is not a homeomorphism but j is continuous, 1–1, and onto, it follows that j cannot be closed. On the other hand, it is known that every compact subset of Y is *finite* (compare [B_2], Exc. I.9.4), and therefore j is proper (in the sense of the definition in these notes).

Proof. ($\Rightarrow$) If f is $\mathcal{A}$-universally closed and $K \subset Y$ is compact, then $K \to \{pt.\}$ is universally closed (Theorem 5), and the map $f_K : f^{-1}(K) \to K$ given by $f|f^{-1}(K)$ is also universally closed (a routine verification). Since the composite of universally closed maps is universally closed (see Lemma 5), it follows that the constant map $f^{-1}(K) \to \{\text{pt.}\}$ is also universally closed. But by Theorem 5 this implies that $f^{-1}(K)$ is compact. □

($\Leftarrow$) If f is proper, then by Theorem 3 we know that $f \times 1_Z$ is proper for all spaces Z (because an identity map is always proper). Therefore by Theorem 1 the map $f \times 1_Z$ is closed.■

When working with $\mathcal{A}$-universally closed maps it is often useful to have a smaller set of test spaces $\mathcal{A}_0$ such that a map is universally closed if and only if its product with the identity map of a space in $\mathcal{A}_0$ is closed. Here is an abstract version and its most important special case(s).

Lemma 8. *Let $\mathcal{A}$ be a family of Hausdorff spaces satisfying the conditions at the beginning of this section, and let $\mathcal{A}_0$ be a subfamily of $\mathcal{A}$ such that every space in $\mathcal{A}$ is homeomorphic to a closed subset of a space in $\mathcal{A}_0$. Let X and Y be spaces in $\mathcal{A}$. Then a continuous map $f : X \to Y$ is $\mathcal{A}$-universally closed if and only if $f \times 1_E$ is closed for all spaces $E \in \mathcal{A}_0$.*

Important note: It is not assumed that X or Y lies in the subfamily $\mathcal{A}_0$.

Proof. The ($\Leftarrow$) implication is immediate. To prove the ($\Rightarrow$) implication, let $Z \in \mathcal{A}$, and let Z be homeomorphic to a closed subspace of E_0. Without loss of generality we can assume that Z is in fact a subspace of E_0. Then $X \times Z$ is a closed subset of $X \times E_0$, and therefore it follows that $f \times 1_{E_0}|X \times Z$ is a closed mapping. But this map factors as $j(f \times 1_Z)$, where j is the closed map defined by the inclusion $X \times Z \subset X \times E_0$. It follows that $f \times 1_Z$ must be a closed map.$\blacksquare$

EXAMPLES. Suppose that $\mathcal{A}$ is the family of all closed subsets of $\mathbb{R}^n$, where n ranges over all positive integers. Then two choices for $\mathcal{A}_0$ are the family $\mathcal{E}_{\mathbb{R}}$ consisting of all the spaces $\mathbb{R}^n$ and the family $\mathcal{E}_{\mathbb{C}}$ consisting of all the spaces $\mathbb{C}^n$.

Theorem 9. *Let $\mathcal{A}$ be the family of all closed subsets of $\mathbb{R}^n$, where n ranges over all positive integers, let $X, Y \in \mathcal{A}$, and let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Then a continuous map $f : X \to Y$ is proper if and only if $f \times 1_{\mathbb{F}^n}$ is closed for all positive integers n.*

Proof. By Theorem 6 the map f is proper if and only if it is $\mathcal{A}$-universally closed, and by Lemma 8 this is true if and only if $f \times 1_E$ is closed for all $E \in \mathcal{E}_{\mathbb{R}}$ or $\mathcal{E}_{\mathbb{C}}$.$\blacksquare$

In algebraic and complex analytic geometry this result is used extensively (compare [B_1]).

Exercises

1. Suppose that $f : X \to Y$ is a continuous map of noncompact locally compact T_2 spaces. Let $f^\bullet : X^\bullet \to Y^\bullet$ be the map of one point comapactifications defined by $f^\bullet|X = f$ and $f^\bullet(\infty_X) = (\infty_Y)$. Prove that f is proper if and only if $f^\bullet$ is continuous.

2. Prove analogs of Theorems 6 and 7 for $\mathcal{A}$ = the class of all separable metric spaces. [*Hint:* Why is every separable metric space homeomorphic to a subspace of a compact one?]

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