Quotient Spaces

Set-theoretic considerations

Let X be a set, and let R be an equivalence relation on X. Then the set of equivalence classes on x is denoted by X/R, and there is a well-defined map of sets $p : X \to X/R$ sending a point x to the unique equivalence class $[x]$ of R that contains x. By construction this map is onto.

Supppose now that $f : X \to Y$ is an arbitrary onto map of sets, and define an equivalence relation R_f on X by $u \sim v$ if and only if $f(u) = f(v)$. Then there is a canonical 1–1 correspondence between y and X/R; specifically, if we define $h : X/R \to Y$ by $h([x]) = f(x)$, then h is well-defined, 1–1 and onto (verify these!).

Topological structures

Supose now that X has a topology, say $\mathcal{U}$. It is natural to ask if there is a preferred way of imposing a topology on the set of equivalence classes X/R.

The key to doing this is given by looking at functions. In set theory one has the following result:

Proposition. *Let $f : X \to Y$ be a function, let R be an equivalence relation on X, and let $p : X \to X/R$ be the map sending an element to its equivalence class. Suppose that whenever $u \sim_R v$ in X we have $f(u) = f(v)$. Then there is a unique function $g : X/R \to Y$ such that $g \circ p = f$.*

The point is that a well-defined function is obtained from the formula $g([x]) = f(x)$.

Suppose now that $(X, \mathcal{U})$ is a topological space. We want a topology on X/R so that p is continuous and the proposition remains true when "continuous function" replaces "function" in the sense of set theory.

DEFINITION. The *quotient topology $p_*\mathcal{U}$ on X/R* is defined by the condition that W is open in X/R if and only if its inverse image $p^{-1}(W)$ is open in X.

The first thing to check is that this forms a topology on X/R. The inverse image of the empty set is the empty set and the inverse image of X/R is X, so $p_*\mathcal{U}$ contains the empty set and X/R. If U_α lies in X/R for all α, then $p^{-1}(\cup_\alpha U_\alpha) = \cup_\alpha p^{-1}(U_\alpha)$ where each term on the right hand side lies in $\mathcal{U}$ by the definition of $p_*\mathcal{U}$; since the union of open sets in X is again an open subset, it follows that $p^{-1}(\cup_\alpha U_\alpha)$ is open in X which in turn implies that $\cup_\alpha U_\alpha$ belongs to $p_*\mathcal{U}$. Likewise, if U_1 and U_2 are in $p_*\mathcal{U}$, then $p^{-1}(U_1 \cap U_2) = p^{-1}(U_1) \cap p^{-1}(U_2)$, and each term of the right hand side lies in $\mathcal{U}$; since the latter is a topology for x it follows that the right hand side also lies in $\mathcal{U}$, and therefore it follows that $U_1 \cap U_2$ lies in $p_*\mathcal{U}$.

Proof of topological version of the Proposition. The only thing that needs to be checked is the continuity of g. Suppose that W is open in Y. Then $g^{-1}(W)$ is open in X/R if and only if $p^{-1}(g^{-1}(W)$ is open in X. But $g \circ p = f$, and hence $p^{-1}(g^{-1}(W) = f^{-1}(W)$. Since f is continuous, the latter is in fact open in X. Therefore $g^{-1}(W)$ is indeed open, so that g is continuous as required.

Definition. Let $f : X \to Y$ be continuous and onto, let R_f be the equivalence relation described above, and let $h : X/R \to Y$ be the standard 1–1 onto map described above. By construction, h is continuous if X/R is given the quotient topology. We say that f *is a quotient map* if h is a homeomorphism.

By the definitions, f is a quotient map if and only if for every $B \subset Y$ we have that B is open in Y if and only if $f^{-1}(B)$ is open in X. — The statement remains true if one replaces "open" by "closed" everywhere.

Exercise. Prove that the quotient topology is the largest topology on X/R such that $p : X \to X/R$ is continuous.