Math 205C
Spring 2003

Take home assignment 1

Due Wednesday, April 30, 2003

1. Let M be a second countable topological n-manifold that is not compact. Prove that there is a continuous map from M to the real line that is *proper*; in other words, inverse images of compact subsets are compact. [*Hint:* Take a countable locally finite open covering $\{W_j\}$ of M by subsets homeomorphic to the open unit disk in $\mathbf{R}^n$ such that the images $\{U_j\}$ of the disks of radius $\frac{1}{3}$ still form an open covering of M. Note that the closures of the open sets $\{U_j\}$ are all compact. Let φ_j be a continuous function that is 1 on $\{U_j\}$ and 0 off the image of the disk of radius $\frac{2}{3}$ in W_j. Consider the function $f = \Sigma_j j\varphi_j$. Why does it suffice to show that the inverse image of each closed interval $[0, k]$ is compact? If n is a positive integer, why does $x \in U_j$ for $j > n$ imply that $f(x) > n$? Why does the latter imply that $f^1([0, n]) \subset U_1 \cup \cdots U_n$ and that the left hand side is compact?]

2. Prove that there is no smooth 1–1 mapping f from $\mathbf{R}^n$ to $\mathbf{R}$ if $n > 1$. [*Hint:* Why is this impossible if the derivative of f is always zero? If the derivative is nonzero at some point x, why is f a submersion near x? Why does the local description of submersions imply that f cannot be 1–1?]

3. Let U, V and W be open subsets of Euclidean spaces, and suppose that $f : U \to V$ and $g : V \to W$ are smooth submersions. Prove that the composite gf is also a submersion, and prove that f and g are open mappings.

4. If U and V are open subsets of Euclidean spaces then two smooth maps $f, g : U \to V$ are said to be *smoothly homotopic* if there is a continuous map $H : U \times [0, 1] \to V$ such that H is smooth on $U \times (0, 1)$ and there is an $\varepsilon > 0$ such that $H(x, t) = f(x)$ when $t < \varepsilon$ and $H(x, t) = g(x)$ when $t > 1 - \varepsilon$.

(i) Prove that this defines an equivalence relation on the set of all smooth maps from U to V.

(ii) Suppose that U is a convex subset of $\mathbf{R}^n$ (*i.e.*, if x and y belong to U and $t \in [0, 1]$ then $tx + (1 - t)y \in U$). Prove that the identity map is smoothly homotopic to a constant map. [*Hint:* A continuous homotopy is defined by $H(x, t) = tz + (1 - t)x$ for some fixed $z \in U$. How can one use C^∞ bump functions to modify this so that $H(x, t)$ depends only on x for t close to 0 or 1?]

5. Consider the vector field on the plane whose principal part is the linear function $\mathbf{F}(x, y) = (2x, x + 2y)$. Find the integral curves of this vector field and explain why the associated flow defines a smooth map Φ from $\mathbf{R}^2$ to $\mathbf{R}$. If Γ is the unit circle in the Cartesian plane, describe the curve $\Phi(C \times \{1\})$ by means of an algebraic equation in x and y.