Math 205C

Spring 2003

SOLUTIONS TO

Take home assignment 1

Due Wednesday, April 30, 2003

1. Let M be a second countable topological n-manifold that is not compact. Prove that there is a continuous map from M to the real line that is *proper*; in other words, inverse images of compact subsets are compact. [*Hint:* Take a countable locally finite open covering $\{W_j\}$ of M by subsets homeomorphic to the open unit disk in $\mathbf{R}^n$ such that the images $\{U_j\}$ of the disks of radius $\frac{1}{3}$ still form an open covering of M. Note that the closures of the open sets $\{U_j\}$ are all compact. Let φ_j be a continuous function that is 1 on $\{U_j\}$ and 0 off the image of the disk of radius $\frac{2}{3}$ in W_j. Consider the function $f = \Sigma_j j\varphi_j$. Why does it suffice to show that the inverse image of each closed interval $[0, k]$ is compact? If n is a positive integer, why does $x \in U_j$ for $j > n$ imply that $f(x) > n$? Why does the latter imply that $f^1([0, n]) \subset U_1 \cup \cdots U_n$ and that the left hand side is compact?]

Solution. Since M is second countable there is an open covering of the type described and the functions φ_j are constructed in the proof of existence of partitions of unity.

Suppose that the inverse image of each closed interval $[0, k]$ is compact; we claim that the inverse image of every compact subset of the real line is also compact. First of all, since the value of the function is nonnegative everywhere, it suffices to show that the inverse image of every compact subset of the nonnegative reals is also compact. Let K be such a subset; then there is a positive constant k such that $K \subset [0, k]$. We then have that $f^{-1}(K) \subset f^{-1}([0, k])$. The former is closed because K is closed, and the latter is compact by assumption. Since closed subsets of compact subsets are compact, it follows that $f^{-1}(K)$ is compact. Therefore the continuous map f is proper if the inverse image of each closed interval $[0, k]$ is compact.

Suppose now that $x \in U_m$; then $f(x) = \Sigma j\varphi_j(x) \geq m\varphi_m(x)$, and since $x \in U_m$ we have $\varphi_m(x) = 1$ which in turn implies that $f(x) \geq m$. Consequently, if $x \in U_j$ for $j > n$ then $f(x) > n$. Taking contrapositives we see that $f(x) \leq n$ implies $x \notin U_j$ for $j > n$. Since $M = \cup_j U_j$ this means that $x \in U_1 \cup \cdots \cup U_n$ if $f(x) \leq n$. In other words, $f^{-1}([0, n]) \subset U_1 \cup \cdots U_n$.

By hypothesis each set U_j has compact closure, and therefore $\overline{U_1} \cup \cdots \overline{U_n}$ is compact. Since $f^{-1}([0, n])$ is contained in the latter and is a closed subset of M, it follows that $f^{-1}([0, n])$ is compact. Therefore the map f is proper by the reduction in the second paragraph.

2. Prove that there is no smooth 1–1 mapping f from $\mathbf{R}^n$ to $\mathbf{R}$ if $n > 1$. [*Hint:* Why is this impossible if the derivative of f is always zero? If the derivative is nonzero at some point x, why is f a submersion near x? Why does the local description of submersions imply that f cannot be 1–1?]

Solution. If Df is identically zero then f is constant and therefore it is not 1–1. Assume now that $Df(x) \neq 0$ for some x. By continuity of the partial derivatives there is an open neighborhood U of x on which Df is never zero; since the target space for f is 1-dimensional it follows that $Df(y)$ is onto for all $y \in U$, so that $f|U$ is a submersion. Therefore there is an open set V in $\mathbf{R}^{n-1}$ and a diffeomorphism h from $V \times (a, b)$ (for suitable real numbers a and b) to some neighborhood W of x with $W \subset U$ such that $x = h(z, f(x))$ for some z and $f \circ h(v, t) = t$. Since $n > 1$ it follows that neither $f \circ h$ nor f can be 1–1. In fact, the inverse image of $\{f(x)\}$ contains an uncountable set that is homeomorphic to V.

3. Let U, V and W be open subsets of Euclidean spaces, and suppose that $f : U \to V$ and $g : V \to W$ are smooth submersions. Prove that the composite gf is also a submersion, and prove that f and g are open mappings.

Solution. By definition $Df(u)$ and $Dg(v)$ are onto for all u and v. But $Dg \circ f(u) = Dg(u)Df(g(u))$ and therefore is the composite of two linear maps that are onto, which means that it must also be onto. It suffices to prove the second assertion for f (the hypothesis does not involve g, and one can simply use the same proof with changed variables). Let $x \in U$. Since f is a submersion, one can find a neighborhood U_0 of x contained in U, a neighborhood V_0 of $f(x)$ contained in V, an open subset N in a Eucldean space of dimension $\dim U - \dim V$, and a diffeomorphism h from $V_0 \times N$ to U_0 such that $f \circ h(v, z) = v$. Since projection onto a factor of a cartesian product is always an open mapping, it follows that $f \circ h$ is open, and since h is a homeomorphism onto U_0 it also follows that $f|U_0$ is open. Since x was an arbitrary point of U it follows that each point of U has an open neighborhood U_x for which $f|U_x$ is open. Since these open sets form an open covering of U it follows that f itself must be open.

4. If U and V are open subsets of Euclidean spaces then two smooth maps $f, g : U \to V$ are said to be *smoothly homotopic*, written $f \simeq_s g$, if there is a continuous map $H : U \times [0, 1] \to V$ such that H is smooth on $U \times (0, 1)$ and there is an $\varepsilon > 0$ such that $H(x, t) = f(x)$ when $t < \varepsilon$ and $H(x, t) = g(x)$ when $t > 1 - \varepsilon$.

(i) Prove that this defines an equivalence relation on the set of all smooth maps from U to V.

(ii) Suppose that U is a convex subset of $\mathbf{R}^n$ (*i.e.*, if x and y belong to U and $t \in [0, 1]$ then $tx + (1 - t)y \in U$). Prove that the identity map is smoothly homotopic to a constant map. [*Hint:* A continuous homotopy is defined by $H(x, t) = tz + (1 - t)x$ for some fixed $z \in U$. How can one use C^∞ bump functions to modify this so that $H(x, t)$ depends only on x for t close to 0 or 1?]

Solution. (i) To see that a map is homotopic to itself take $H(x, t) = (f(x), t)$ for all x and t. This map is smooth on $U \times (0, 1)$ if f is smooth, and by construction the homotopy depends only on x if t is close to 0 or 1 (in fact, this is true for **ALL** t!). Suppose now that $f \simeq_s g$, and let H be the map described in the definition of this statement. Consider the map $K(x, t) = H(x, 1 - t)$. Then K is smooth on $U \times (0, 1)$, and we have $K(x, t) = g(x)$ for t close to 0 and $K(x, t) = f(x)$ for t close to 1,

Finally, suppose that $f \simeq_s g$ and $g \simeq_s h$. Let K be the continuous map $U \times [0, 1] \to V$ such that K is smooth on $U \times (0, 1)$ and there is an $\varepsilon > 0$ such that $K(x, t) = f(x)$ when $t < \varepsilon$ and $K(x, t) = g(x)$ when $g(x) > 1 - \varepsilon$, and let L be the continuous map $U \times [0, 1] \to V$ such that L is smooth on $U \times (0, 1)$ and there is a $\delta > 0$ such that $L(x, t) = g(x)$ when $t < \delta$ and $L(x, t) = h(x)$ when $t > 1 - \delta$. As in the continuous case, define a continuous map $Q : U \times [0, 1] \to V$ by $Q(x, t) = K(x, 2t)$ if $t \le \frac{1}{2}$ and $Q(x, t) = L(x, 2t - 1)$ if $t \ge \frac{1}{2}$. By construction it follows that $Q(x, t) = f(x)$ if t is close to 0 and $Q(x, t) = h(x)$ if t is close to 1. It remains to show that Q is smooth on $U \times (0, 1)$.

It suffices to show that the restrictions of Q to subsets of some open covering for $U \times (0, 1)$ are smooth. By construction the restrictions to $U \times (0, \frac{1}{2})$ and $U \times (\frac{1}{2}, 1)$ are smooth. It remains to show that Q is smooth on an open neighborhood of $U \times \{\frac{1}{2}\}$. However, by construction $Q(x, t) = g(x)$ if $t \in (\frac{1-\varepsilon}{2}, \frac{1}{2}]$ (by the conditions on K) and $Q(x, t) = g(x)$ if $t \in [\frac{1}{2}, \frac{1+\delta}{2})$ (by the conditions on L). It follows that $Q(x, t) = g(x)$ on the open set

$$U \times \left(\frac{1 - \varepsilon}{2}, \frac{1 + \delta}{2} \right)$$

and therefore is smooth on the latter. This completes the proof that Q is smooth on $U \times (0, 1)$.

Solution. (*ii*) Let $\alpha : \mathbf{R} \to \mathbf{R}$ be a smooth map such that $\alpha(t) = 0$ for $t \leq \frac{1}{4}$, α is increasing between $\frac{1}{4}$ and $\frac{3}{4}$, and $\alpha(t) = 1$ for $t \geq \frac{3}{4}$. Modify the straight line homotopy by writing

$$K(x,t) = \alpha(t)z + (1 - \alpha(t))x.$$

It follows that K is smooth on $U \times (0,1)$, while $K(x,t) = x$ for t close to 0 and $K(x,t)$ takes the constant value z for t close to 1.

5. Consider the vector field on the plane whose principal part is the linear function $\mathbf{F}(x,y) = (2x, x+2y)$. Find the integral curves of this vector field and explain why the associated flow defines a smooth map Φ from $\mathbf{R}^2$ to $\mathbf{R}$. If Γ is the unit circle in the Cartesian plane, describe the curve $\Phi(C \times \{1\})$ by means of an algebraic equation in x and y.

Solution. The function $\mathbf{F}$ corresponds to the linear transformation associated to the matrix

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}.$$

We can write $A = 2I + N$ where N is the nilpotent matrix

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

which satisfies $N^2 = 0$. The integral curves of this vector field have the form $\exp(tA)V$ where V is an arbitrary element of $\mathbf{R}^2$ and the latter is identified with the space of all 2×1 matrices.

The first step is to compute $\exp(tA)$ using the basic rules for manipulating such expressions:

$$\exp(tA) = \exp(2tI + tN) = \exp(2tI)\exp(tN) = e^{2t}(I + tN)$$

This gives us an explicit formula for the exponential matrix:

$$\exp(tA) = \begin{pmatrix} e^{2t} & 0 \\ te^{2t} & e^{2t} \end{pmatrix}$$

and the latter in turn yields the formula

$$\varphi_t(x,y) = (e^{2t}x, te^{2t}x + e^{2t}y).$$

Set the coordinates of the right hand side equal to u and v respectively. We need to find the functions A and B such that $x = A(u,v)$ and $y = B(u,v)$. The equation defining the image of the unit circle will then have the form $A(u,v)^2 + B(u,v)^2 = 1$.

In this situation u and v are linear functions of x and y, so solving for x and y is essentially an exercise in elementary algebra. The solutions are

$$x = e^{-2t}u, \qquad y = e^{-2t}v - te^{-2t}u$$

and if we set $t = 1$ we obtain the equation

$$1 = x^2 + y^2 = e^{-4}u^2 + e^{-4}(v-u)^2$$

which simplifies to the following equation that defines an ellipse:

$$1 = e^{-4} \cdot \left(2u^2 - 2uv + v^2\right)$$

The final step is to replace u and v with x and y.

Obviously one can use matrix algebra to describe this ellipse very explicitly, but this will not be done here.