Hints for Take Home Assignment 2

III.21. Let $h_1, h_2 : \mathbf{R}^2$ be stereographic projection charts for S^2 covering the complements of the south and north poles respectively, and let $\psi : \mathbf{R}^2 - \{0\} \to \mathbf{R}^2 - \{0\}$ be the transition diffeomorphism "$h_2^{-1} \circ h_1$" for these charts. Then

$$\psi(x,y) = (u(x,y), v(x,y)) = \left(\frac{4x}{x^2+y^2}\, , \, \frac{4y}{x^2+y^2} \right).$$

It follows that

$$D\psi(x,y) \quad = \quad 4 \cdot \begin{pmatrix} \frac{y^2-x^2}{(x^2+y^2)^2} & \frac{2xy}{(x^2+y^2)^2} \\ \frac{2xy}{(x^2+y^2)^2} & \frac{x^2-y^2}{(x^2+y^2)^2} \end{pmatrix}.$$

If X is the vector field on U_1 given by $\frac{\partial}{\partial x}$, show that $\psi_*(X|U_1 - \{0\})$ is equal to

$$Y(u,v) = \frac{1}{4}(v^2 - u^2)\frac{\partial}{\partial u} + \frac{1}{2}(uv)\frac{\partial}{\partial v}$$

as a vector field on $U_2 - \{0\}$. Why can one now extend Y to a smooth vector field on U_2 by setting $Y(0) = 0$?

Conlon, 4.1.8, p. 133

We are given a smooth vector field X on a manifold M such that $X(p) \neq 0$ and the flow curve $\varphi_t(p)$ for X with initial condition p satisfies $\varphi_c(p) = p$ for some $c > 0$. We are to prove that there is a $b > 0$ so that $\varphi_t(p) = \gamma(\exp(2\pi i t/b)$ for some closed curve $\gamma : S^1 \to M$.

It suffices to show that $\varphi_t(p)$ is defined for all t and that $\varphi_{t+b}(p) = \varphi_t(p)$ for all t and a suitable $b > 0$. By hypothesis one can find $\varepsilon > 0$ so that the curve is defined on $(-\varepsilon, c + \varepsilon)$. Why is $\beta_t = \varphi_{t+c}(p)$ an integral curve defined on $(-c - \varepsilon, \varepsilon)$ with initial condition p, and why does this mean that $\varphi_t(p)$ is defined on $(-c - \varepsilon, c + \varepsilon)$? If we define $\alpha_t = \varphi_{t-c}(p)$ on $(-\varepsilon, 2c + \varepsilon)$, why do α_t and $\varphi_t(p)$ fit together to extend the integral curve to $(-c-\varepsilon, 2c+\varepsilon)$. By construction this extended curve satisfies $\varphi t \pm c(p) = \varphi_t(p)$ for all t such that $t \pm c$ makes sense. How can one continue this process to define the integral curve for all values of c such that $\varphi_{t+b}(p) = \varphi_t(p)$?

Note. It turns out that there is a minimum positive real number b such that $\varphi_{t+b}(p) = \varphi_t(p)$ for all t and if a is any other positive number such that $\varphi_{t+a}(p) = \varphi_t(p)$ for all t then a is a positive integral multiple of b.

VI.6. This requires the following basic fact about exterior powers:

Lemma. *let V be a finite dimensional vector space, and let $0 \neq v \in V$. For each k such that $0 \leq k < \dim V$, define L_v to be the linear map from $\wedge^k(V) \to \wedge^{k+1}(V)$ sending x to $v \wedge x$. Then $L_v(x) = 0$ if and only if $x = L_v(y)$ for some y.*

Proof. Since $v \wedge v \wedge y = 0$ it follows that $L_v(x) = 0$ if $x = L_v(y)$. Suppose now that $v \wedge x = 0$. Pick a basis $\{v_1, \cdots, v_n\}$ for V such that $v = v_1$. Then $L_v(x) = 0$ if and only if x is a linear combination of the associated basis vectors for $\wedge^k(V)$ having the form

$$v_1 \wedge v_{i_2} \wedge \cdots \wedge v_{i_k}$$

where $1 < i_2 < \cdots < i_k$. But this set is also a basis for the image of the linear transformation L_v from $\wedge^{k-1}(V)$ to $\wedge^k(V)$.

Application to the problem. Following the hint, we begin by looking at the local situation where U is open in $\mathbf{R}^n$. By hypothesis ω is a nowhere zero 1-form, so we may write

$$\omega = \sum_i F^i \, dx^i$$

where the smooth functions F^i do not vanish simultaneously. Since we are looking locally we may as well restrict to a neighborhood of a point on which one of the functions is never zero, and by permuting the coordinates if necessary we may assume "without loss of generality" that F^1 is never zero on our open set U. This means that at each point p in the open set the forms $\omega, dx^2, \cdots, dx^n$ define a basis for the cotangent space at p.

Explain why every 2-form θ on U can be written in the form

$$\omega \wedge \lambda + \sum_{2 \leq i < j \leq n} h^{i,j} \, dx^i \wedge dx^j$$

where $\lambda \in \wedge^1(U)$ and $h^{i,j} \in C^\infty(U)$. We shall assume that $n \geq 3$ to keep the discussion nontrivial.

Suppose now that $\omega \wedge \theta = 0$. Explain why $h^{i,j} = 0$ for all i, j. This proves the local statement because we can take $\alpha = -\lambda$.

To prove the global statement, assume one has $d\omega = \eta_V \wedge \omega$ for all V in some nice open covering, and take a subordinate partition of unity $\{\varphi_V\}$ Then each form $\varphi_V \eta_V$ extends to a form θ_V on M by taking it to be zero off V. Why and how can we take α to be $\Sigma_V \theta_V$?

VI.20. The first part can be checked locally by a direct calculation or by the property

$$\beta \wedge \alpha = (-1)^{pq} \alpha \wedge \beta$$

where α is a p-form and β is a q-form. To prove the second part on 2-forms it is instructive to look at the case $n = 2$ and a typical 2-form Ω on $\mathbf{R}^4$:

$$A \, dx^1 \wedge dx^2 + B \, dx^1 \wedge dx^3 + C \, dx^1 \wedge dx^4 + D \, dx^2 \wedge dx^3 + E \, dx^2 \wedge dx^4 + F \, dx^3 \wedge dx^4$$

Under what conditions is $\Omega \wedge \Omega \neq 0$? How can one do this with exactly two of A, B, C, D, E, F taken to be nonzero constants? Specifically, how can this be done if we set $A = 1$? How can this be generalized to dimension 6 and higher?