TANGENT BUNDLES AND RELATED CONSTRUCTIONS

Mathematics 205C, Spring 2003

1. Motivation and goals

A basic idea underlying the theory of smooth manifolds is that such objects can be studied using a mixture of techniques from multivariable calculus and point set topology. We have already discussed some constructions for topological spaces for which there are similar constructions on smooth manifolds in at least some cases, including finite products, covering space projections, submanifolds, quotient constructions related to covering space projections and disjoint sums.

On the other hand, there are also clear differences between what one can do for topological spaces as opposed to smooth manifolds. In particular, there are numerous constructions on topological spaces that do not work at all for smooth manifolds. On the other hand, there are also some important constructions for smooth manifolds that cannot be carried out for topological spaces. The **tangent bundle** of a smooth manifold is a fundamental example of this sort.

The definition of the tangent bundle requires some digressions, so it seems best to begin with a description of what we want. For an open subsubset U of $\mathbf{R}^n$ we defined the space of all tangent vectors to points of U to be the product $U \times \mathbf{R}^n$, the idea being that for each $x \in U$ the space $\{x\} \times \mathbf{R}^n$ can be viewed as a space of tangent vectors at x (or as a a physicist might say, vectors whose point of application is x). Similarly, if we are given a smooth n-manifold $(M, \mathcal{A})$ and a point p in M, we would like to describe a smooth manifold $T(M)$ such that for each $p \in M$ it contains an n-dimensional vector space $T_p(M)$ of *tangent vectors* to p in M, and such that $T(M)$ is the union of these vector spaces for all $p \in M$; for the record, we would also like these vector spaces to be pairwise disjoint. The space $T_p(M)$ should be defined so that its elements can be viewed as tangent vectors for smooth curves $\varphi : (-\varepsilon, \varepsilon)$ satisfying $\varphi(0) = p$; in other words, for each vector $\mathbf{v} \in T_p(M)$ one can find a φ of this sort so that it makes sense to say $\varphi'(0) = \mathbf{v}$.

If M is open in $\mathbf{R}^n$ our previous construction fulfills these requirements. Perhaps the best test case for extending the definition is the standard 2-sphere in Euclidean 3-space.

There are two possible approaches, and they lead to the same answer. On one hand, in classical solid geometry one speaks about the tangent plane to a point on a sphere as the plane perpendicular to the radius at the point of contact. This is good for looking at a single tangent plane, but classical tangent planes generally intersect in a line and we want our tangent planes at different points to be pairwise distinct. One way of creating an object that fulfills this requirements and still leads to the classical notion of tangent plane is to view the tangent space for S^2 to be the set of all points $(x, v) \in S^2 \times \mathbf{R}^3$ such that $|x| = 1$ (*i.e.*, it lies on S^2) and y is perpendicular to x. The classical tangent plane to x is then the set of all points of the form $x + y$ where y is perpendicular to x.

A second way of approaching this is through the following elementary result:

Proposition. *Let $x \in S^2$ and $y \in \mathbf{R}^3$. Then there is a smooth curve $\varphi : (-\varepsilon, \varepsilon) \to S^2$ such that $\varphi(0) = x$ and $\varphi'(0) = y$ if and only if $\langle x, y \rangle = 0$, where $\langle\ ,\ \rangle$ denotes the usual inner product.*

In fact, this all generalizes to level sets of regular values. If $f : \mathbf{R}^n \to \mathbf{R}^m$ is a smooth map (where $n > m$ as usual) and y is a nontrivial regular value of f, then the tangent space of level set $L = f^{-1}(\{y\})$ can be taken to be the set of all points $(u, \mathbf{v}) \in L \times \mathbf{R}^n$ such that $Df(u)\mathbf{v} = 0$. Since f is a regular value the dimension of the kernel of $Df(u)$ is $n - m$ for all u. The preceding proposition extends directly to such level sets with this definition of the tangent space. In particular, for the unit sphere we are looking at the set of all points where $f(x) = 1$, where $f(x) = |x|^2$, and in this case $Df(x)y = 2\langle x, y\rangle$.

By the results in the notes, *Level sets of regular values* and *Level sets of regular values – II*, there is an atlas of smooth charts (U_α, h_α) for L such that each $j \circ h_\alpha$ is smooth. Suppose that $\in L$ is chosen so that $x \in h_\alpha(U_\alpha) \cap h_\beta(U_\beta)$, and let $\mathbf{v}$ be a vector in the kernel of $Df(x)$. Then one can use the coordinate charts to construct smooth curves $\Gamma_\alpha : (-\varepsilon, \varepsilon) \to U_\alpha$ and $\Gamma_\beta : (-\varepsilon, \varepsilon) \to U_\beta$ such that $h_\alpha \circ \Gamma_\alpha = h_\beta \circ \Gamma_\beta$, $h_\alpha(\Gamma_\alpha(0)) = h_\beta(\Gamma_\beta(0))$ and if $\Gamma : (-\varepsilon, \varepsilon) \to \mathbf{R}^n$ is the the associated smooth curve in Euclidean m-space then $\Gamma'(0) = \mathbf{v}$.

FUNDAMENTAL QUESTION. *What is the relationship between the tangent vectors $\Gamma'_\alpha(0)$ and $\Gamma'_\beta(0)$?*

ANSWER. By construction we have that Γ_β is equal to "$h_\beta^{-1}h_\alpha$" $\circ\, \Gamma_\alpha$, and therefore by the Chain Rule the tangent vector $\mathbf{w}$ at $u = \Gamma_\alpha(0)$ is identified with the tangent vector D"$h_\beta^{-1}h_\alpha$"$(u)\mathbf{w}$ at "$h_\beta^{-1}h_\alpha$"$(u) = \Gamma_\beta(0)$.

All of these considerations are part of the following result:

THEOREM. *Let $n > m$ and let $f : \mathbf{R}^n \to \mathbf{R}^m$ be a smooth map such that y is a nontrivial regular value f (i.e., there is some x so that $f(x) = y$), and let $L = f^{-1}(\{y\})$. Then the tangent space to L, consisting of all $(x, y) \in L \times \mathbf{R}^n$ such that $Df(x)y = 0$, is a smooth manifold, and if $\mathcal{A} = \{(U_\alpha, h_\alpha)\}$ is a smooth atlas of the type described above, then there is a smooth atlas for the tangent space of L having the form $\{(U_\alpha \times \mathbf{R}^{n-m}, H_\alpha)\}$ where $H_\alpha(x, \mathbf{v}) = (h_\alpha(x), Dh_\alpha(x)\mathbf{v})$.*

The transition maps are smooth because they are given by the formula "$H_\beta^{-1}H_\alpha$"$(x, \mathbf{v}) =$ ("$h_\beta^{-1}h_\alpha$" $(x), D[$"$h_\beta^{-1}h_\alpha$"$](x)\mathbf{v})$.

2. General construction for the tangent bundle

We would like to have a similar construction for the tangent bundle of a smooth n-manifold M. Specifically, given an atlas $\mathcal{A} = \{(U_\alpha, h_\alpha)\}$ for M we would like to have a comparable atlas $T(\mathcal{A})$ of the form $\{(U_\alpha \times \mathbf{R}^n, H_\alpha)\}$ for the tangent space $T(M)$. Although we do yet know how to construct the space $T(M)$ or the coordinate charts H_α, the formula at the end of the last section makes sense and provides a description of the transition maps for the charts in the desired atlas.

The results described in the second part of the note, *Constructing topological spaces out of pieces*, provide a method for constructing the space we want. We begin by recalling the formal definition.

Definition. A *set of topological amalgamation data* is a pair

$$(\{Y_\alpha\}, \{\varphi_{\beta\alpha}\})$$

where $\{Y_\alpha\}$ is an indexed family of topological spaces with indexing set $\mathbf{A}$ and $\{\varphi_{\beta\alpha}\}$ is an indexed family of homeomorphisms with indexing set $\mathbf{A} \times \mathbf{A}$ such that the following conditions hold:

(i) For every α and β the map $\varphi_{\beta\alpha}$ is a homeomorphism from an open subset $W_{\beta\alpha}$ of Y_α to an open subset $W_{\alpha\beta}$ of Y_β.

(ii) For every α the map $\varphi_{\alpha\alpha}$ is the identity map for Y_α, and for every α and β the map $\varphi_{\alpha\beta}$ is the inverse homeomorphism to $\varphi_{\beta\alpha}$.

(iii) For every α, β and γ the map $\varphi_{\beta\alpha}$ sends the open set $W_{\beta\alpha} \cap W_{\gamma\alpha} \subset Y_\alpha$ homeomorphically onto $W_{\alpha\beta} \cap W_{\gamma\beta} \subset Y_\beta$, and $\varphi_{\gamma\beta}(\varphi_{\beta\alpha}(y)) = \varphi_{\gamma\alpha}(y)$ for all $y \in W_{\beta\alpha} \cap W_{\gamma\alpha}$.

The relations described above are called **cocycle formulas** or something similar in Conlon's book and numerous other places (the key word is "cocycle").

The following result was discussed in the previously mentioned notes:

Theorem. *If* $\mathbf{Y} = (\{Y_\alpha\}, \{\varphi_{\beta\alpha}\})$ *is a set of topological amalgamation data then there is a space* X *and an open covering* $\mathcal{U}$ *of* X *with the same indexing set as* $\{Y_\alpha\}$ *such that there are homeomorphisms* $k_\alpha : Y_\alpha \to U_\alpha$ *and*

$$k_\alpha(Y_\alpha) \cap k_\beta(Y_\beta) = k_\alpha(\varphi_{\alpha\beta}(W_{\alpha\beta})) = k_\beta(\varphi_{\beta\alpha}(W_{\beta\alpha}))$$

for all α *and* β*. The space* X *is uniquely determined up to homeomorphism.*

For smooth manifolds one can also give a condition for realizing the amalgamation data by a smooth atlas.

Theorem. *In the preceding result, suppose that the spaces* Y_α *are all open subsets of* $\mathbf{R}^n$*, the maps* $\varphi_{\beta\alpha}$ *are all diffeomorphisms, and the associated space* X *is Hausdorff and second countable. Then* X *is a second countable topological* n*-manifold, and it has a smooth atlas* $\{(U_\alpha, h_\alpha)\}$ *such that the transition maps "*$h_\beta^{-1}h_\alpha$*" are equal to* $\varphi_{\beta\alpha}$ *for all* α *and* β*.*

In the situation of this result, if $\mathcal{A}$ is the original set of amalgamation data then the corresponding smooth atlas on the constructed space X will be called the *associated smooth atlas* for the amalgamation data.

There is a standard example to show that the space constructed in this manner is not necessarily Hausdorff even in the second case. Take the amalgamation data for which $Y_1 = Y_2 = \mathbf{R}^n$, with $W_{2,1} = W_{1,2} = \mathbf{R}^n - \{0\}$ and $\varphi_{2,1} = \varphi_{1,2}$ is the identity. Then the images of the two zero points in X (coming from the zero points of Y_1 and Y_2) do not have disjoint neighborhoods. (*Question:* Is X always a $\mathbf{T}_1$ space? Prove this or give a counterexample.).

Specialization to the tangent bundle

We begin with a smooth atlas $\mathcal{A} = \{(U_\alpha, h_\alpha)\}$ for M and form $T_0(\mathcal{A})$ as indicated before, with candidates for amalgamation data corresponding to the spaces $Y_\alpha = U_\alpha \times \mathbf{R}^n$, with open subspaces $W_{\beta\alpha} = h_\alpha^{-1}(h_\beta(U_\beta)) \times \mathbf{R}^n$ and diffeomorphisms $\Psi_{\beta\alpha}$ sending $(x, \mathbf{v})$ to $(x, D\text{"}h_\beta^{-1}h_\alpha\text{"}(x)\mathbf{v})$. In order to show that we have a set of amalgamation data it is necessary to check that the cocycle formulas hold.

(i) $\Psi_{\alpha\alpha}$ is the identity because "$h_\alpha^{-1}h_\alpha$" is the identity and the derivative of an identity map is just the identity matrix.

(ii) To see that $\Psi_{\beta\alpha}$ and $\Psi_{\alpha\beta}$ are inverse to each other, it suffices to calculate the composites explicitly using the fact that $[\text{"}h_\beta^{-1}h_\alpha\text{"}]^{-1}$ equals "$h_\alpha^{-1}h_\beta$" implies $[D\text{"}h_\beta^{-1}h_\alpha\text{"}](x)^{-1}$ equals $D\text{"}h_\alpha^{-1}h_\beta\text{"}(x)$ for all x.

(iii) To see the composition relation, it suffices to calculate the composites explicitly using the fact that D"$h_\gamma^{-1}h_\alpha$" is the matrix product of equals D"$h_\gamma^{-1}h_\beta$" and D"$h_\beta^{-1}h_\alpha$" by the Chain Rule.

By construction the maps $\Psi_{\beta\alpha}$ are all diffeomorphisms, so we are reduced to showing two things; namely, the space $T(M)$ constructed from the preceding amalgamation data is second countable if M is, and it is always Hausdorff.

The key to doing this involves a fundamental property of the tangent space: The existence of a continuous (in fact, smooth) map $\tau_M : T(M) \to M$ such that for each $p \in M$ the set $\tau_M^{-1}(\{p\}$ is the n-dimensional space of tangent vectors at p and for each smooth chart (U_α, h_α) in $\mathcal{A}$ the corresponding chart $(U_\alpha \times \mathbf{R}^n, H_\alpha)$ satisfies Image$(H_\alpha) = \tau_M^{-1}(h_\alpha(U_\alpha))$ and $\tau_M \circ H_\alpha(x, \mathbf{v}) = h_\alpha(x)$.

The formula at the end of the last paragraph tells us how to construct τ_M on each element of the open covering, and a continuous function is obtained in this way if and only if the stated value is independent of the choice of α. In our situation this amounts to knowing that the first coordinate of $\Psi_{\beta\alpha}(x, \mathbf{v})$ is just "$h_\beta^{-1}h_\alpha$"(x); but this is the definition of the first coordinate of the given mapping. Therefore we have a well-defined continuous map τ_M with the second property. By the construction it is clear that the inverse image of $h_\alpha(U_\alpha)$ contains the open set $H_\alpha(U_\alpha \times \mathbf{R}^n)$; we claim it contains nothing else. Suppose that $H_\beta(y, \mathbf{w})$ maps into $h_\alpha(U_\alpha)$. It then follows that $h_\beta(y) = h_\alpha(x)$ for some $x \in U_\alpha$, and by construction it then also follows that $H_\beta(y, \mathbf{w}) = H_\alpha(x, \mathbf{v})$ for a suitably chosen $\mathbf{v}$; specifically, we choose the latter so that $\Psi_{\alpha\beta}(y, \mathbf{w}) = (x, \mathbf{v})$.

How does this help with the statements we wish to prove? We combine the observations above with the following two exercises in point set topology:

Proposition. *Let X and Y be topological spaces, and let $g : X \to Y$ be a continuous map such that each point $y \in Y$ has an open neighborhood V for which $g^{-1}(V)$ is homeomorphic to a product $V \times F$, for some space F, by a homeomorphism $h : V \times F \to g^{-1}(V)$ satisfying $g(h(v, z)) = v$ for all v and z.*

(A) *If Y and F are second countable then so is X.*

(B) *If Y and F are both Hausdorff then so is X.*

Proof. (*Sketch*) (A) There is a countable open covering $\{V_j\}$ where the open sets satisfy the local hypothesis (Why?). Each of the open subsets is also second countable, and a product of second countable sets is second countable, so X is a countable union of the second countable spaces $g^{-1}(V_j)$. But if a space can be expressed as a countable union of second countable open subsets, it must also be second countable (Why?).

(B) Suppose that $x_1 \neq x_2$ in X. If $g(x_1) \neq g(x_2)$ then there are disjoint neighborhoods U_1 and U_2 of these image points in Y, and the inverse images $g^{-1}(U_1)$ and $g^{-1}(U_2)$ must be disjoint neighborhoods of x_1 and x_2. On the other hand, if $g(x_1) = g(x_2)$ let V be an open neighborhood of this point as described in the hypothesis of the theorem. The inverse image of this neighborhood is homeomorphic to $V \times F$ for some Hausdorff space F, and under this homeomorphism x_i corresponds to (v_i, z_i) where $v_1 = v_2$ but $z_1 \neq z_2$. Choose disjoint neighborhoods W_1 and W_2 for z_1 and z_2 in F such that $z_i \in W_i$ for $i = 1, 2$. Then the images $h(V \times W_1)$ and $h(V \times W_2)$ are open subsets of $g^{-1}(V)$ that are disjoint neighborhoods of x_1 and x_2 in X.

Additional properties of the tangent bundle

Strictly speaking our construction for tangent spaces depends upon choosing a smooth atlas for M. One would expect that equivalent smooth atlases for a manifold yield equivalent tangent spaces. Let us formulate this right now.

Proposition. *Let $\mathcal{A}$ be a smooth atlas for the n-manifold M, and let $\mathcal{M}$ be the associated maximal atlas. Denote the constructions for the tangent space associated to $\mathcal{A}$ and $\mathcal{M}$ by $T(M;\mathcal{A})$ and $T(M;\mathcal{M})$ respectively. Then there is a canonical diffeomorphism $\Phi : T(M;\mathcal{A}) \to T(M;\mathcal{M})$ such that Φ sends the standard smooth charts for $T(M;\mathcal{A})$ to the corresponding standard smooth charts for $T(M;\mathcal{M})$. and $\tau_{(M,\mathcal{M})} \circ \Phi = \tau_{(M,\mathcal{A})}$.*

The proof is relatively elementary but a bit tedious and is left to the reader as an exercise.

Note. The constructed atlas for $T(M;\mathcal{M})$ is not a maximal atlas for the tangent space. Consider the case $M = \mathbf{R}^n$. If we take $\mathcal{A}$ to be the atlas whose only chart is the identity map, then we see that $T(M) \cong M \times \mathbf{R}^n$ such that τ_M corresponds to projection onto the first factor (use the proposition). The charts in $T(M;\mathcal{M})$ all map onto vertical strips of the form $W \times \mathbf{R}^n$, and of course there are many smooth charts on $T(M) \cong M \times \mathbf{R}^n \cong \mathbf{R}^{2n}$ that do not have this form.

We can now state the **formal definition of the tangent space**: Given a smooth manifold $(M,\mathcal{A})$ with $\mathcal{A} = (U_\alpha, h_\alpha)$, the associated manifold $T(M)$ is the smooth manifold associated to the amalgamation data $T_0(\mathcal{A})$:

- $Y_\alpha = U_\alpha \times \mathbf{R}^n$.
- $W_{\beta\alpha} = h_\alpha^{-1}(h_\beta(U_\beta)) \times \mathbf{R}^n$.
- $\Psi_{\beta\alpha}$ is defined so that $(x,\mathbf{v})$ is sent to $(x,\ D\text{“}h_\beta^{-1}h_\alpha\text{”}(x)\mathbf{v})$.

The smooth atlas for $T(M)$ associated to $T_0(\mathcal{A})$ is denoted by $T(\mathcal{A})$ and the unique maximal atlas containing the latter will be denoted by $\widehat{T}(\mathcal{A})$.

The preceding discussion shows that one can in fact construct the tangent space using an arbitrary subatlas of the maximal atlas for M. If, say, we can find a subatlas that is finite, this will clearly make it much easier to study the structure of the tangent space.

Before proceeding we shall verify a parenthetical remark made earlier in this section.

Proposition. *Given a smooth manifold $(M,\mathcal{A})$ let $\tau_M : T(M) \to M$ be the continuous map described above; specifically, for each smooth chart (U_α, h_α) in $\mathcal{A}$ and corresponding chart $(U_\alpha \times \mathbf{R}^n, H_\alpha)$ in $T(\mathcal{A})$ we have $\tau_M \circ H_\alpha(x,\mathbf{v}) = h_\alpha(x)$. Then τ_M is smooth.*

Proof. Given topological spaces A and B let $P_1 : A \times B \to A$ denote projection onto the first factor. By the formula stated in the proposition we know that the image of $\tau_M \circ H_\alpha$ is contained in the image of h_α, and in fact "$h_\alpha^{-1} \circ \tau_M \circ H_\alpha$"$= P_1$. Since P_1 is smooth if A and B are open subsets of $\mathbf{R}^n$, this proves the smoothness of τ_M.

3. Vector space algebra in the tangent bundle

Now that we have constructed the tangent bundle, we need to show that it can be viewed as a union of n-dimensional vector spaces, with one for each point in the manifold. The first step is to notice that for each $p \in M$ the inverse image $T_p(M) = \tau_M^{-1}(\{p\})$ is diffeomorphic to $\mathbf{R}^n$. How and why can we view this as an n-dimensional vector space? As usual take the maximal atlas $\mathcal{A}$, take a chart (U_α, h_α) in $\mathcal{A}$ so that $p = h_\alpha(x)$, and let $(U_\alpha \times \mathbf{R}^n, H_\alpha)$ be the corresponding chart in the smooth atlas $T(\mathcal{A})$. Then we can make $T_p(M)$ into an n-dimensional vector space by brute force if we simply decree that the addition and scalar multiplication are defined so that the 1–1 correspondence from $\{x\} \times \mathbf{R}^n$ to $T_p(M)$ defined by H_α should be an isomorphism of vector spaces.

Once again we need to verify that this does not depend upon the choice of α in order say that the vector space operations are well defined. Suppose now that we have another chart (U_β, h_β) so that $h_\beta(y) = p$. We then obtain a possibly different vector space structure from H_β. To compare these structures we need to consider the vector space structure on $\{y\} \times \mathbf{R}^n$ determined by the usual vector space structure on $\{x\} \times \mathbf{R}^n$ and the 1–1 correspondence between these spaces associated to the map "$H_\beta^{-1} H_\alpha$" from the second to the first. **The new vector space structure on will agree with the usual one if and only if the $1-1$ correspondence is an isomorphism of vector spaces.** To check this, note that the formulas for the transition map "$H_\beta^{-1} H_\alpha$" specialize to "$H_\beta^{-1} H_\alpha$"$(x, \mathbf{v}) = (y, L\mathbf{v})$, where L is the linear isomorphism D"$h_\beta^{-1} h_\alpha$"(x). This implies that the vector space structure on $T(M)$ is well-defined.

We actually need something stronger; namely, that the maps involved in the definition of vector space depend smoothly upon p in some sense. The first step is to establish continuity.

Continuity of the zero map. We claim there is a map $z : M \to T(M)$ such that for each $p \in M$ the tangent vector $z(p)$ is the zero element of the vector space $T_p(M)$. The idea behind the construction is easy: As before, write $p = h_\alpha(x)$, define $z_\alpha(u) = (u, \mathbf{0})$ for $u \in U_\alpha$, and note that this is well defined because the second coordinate of "$H_\beta^{-1} H_\alpha$"$(u, \mathbf{0})$ is equal to D"$H_\beta^{-1} H_\alpha$"$(u)\mathbf{0} = \mathbf{0}$. The continuity of the map follows immediately from the local formula.

Continuity of scalar multiplication. We know there is a set-theoretic map $\mu : \mathbf{R} \times T(M) \to T(M)$ that sends $\mathbf{R} \times T_p(M)$ to $T_p(M)$, for each $p \in M$, by scalar multiplication. In fact, it is given locally by maps μ_α for each $\mathbf{R} \times U_\alpha \times \mathbf{R}^n$ satisfying the formulas $\mu_\alpha(t;\, u, \mathbf{v}) = (u, t\mathbf{v})$. These will yield a continuous map if the construction is well defined, and the verification of this proceeds as for the zero map (in this case using the fact that $L(t\mathbf{v}) = tL(\mathbf{v})$ rather than $L(\mathbf{0}) = \mathbf{0}$.

Continuity of addition. This is more complicated because addition is a function of two vector variables rather than one and we can only add two vectors if they lie over the same point in M. Consider the space of all ordered pairs of tangent vectors that lie over the same point in M:

$$T(M) \times_M T(M) := \{(\mathbf{v}, \mathbf{w}) \in T(M) \times T(M) \mid \tau_M(\mathbf{v}) = \tau_M(\mathbf{w})$$

Vector addition then defines a set-theoretic map Σ from $T(M) \times_M T(M)$ to $T(M)$, and we would like to show that this map is continuous. Once again the approach is to look locally.

There is a well-defined continuous map ${}_2\tau$ from $T(M) \times_M T(M)$ to $T(M)$ that sends $(\mathbf{v}, \mathbf{w})$ to $\tau_M(\mathbf{v}) = \tau_M(\mathbf{w})$. Given a typical chart (U_α, h_α) for M we need to consider ${}_2\tau^{-1}(h_\alpha(U_\alpha)$. If as usual H_α comes from the chart for $T(M)$ corresponding to the original chart for M, then

$$(H_\alpha \times H_\alpha)^{-1}\,(T(M) \times_M T(M)) = \{(u, \mathbf{v};\, z, \mathbf{w}) \mid u = z\}$$

and on this set addition is defined locally by

$$\Sigma_\alpha(u, \mathbf{v};\, u, \mathbf{w}) = (u;\, \mathbf{v}{+}\mathbf{w}).$$

Once again the global continuity of this construction is automatic if the latter is well-defined, and we already know this because each of the derivatives $D``h_\beta^{-1}h_\alpha"(u)$ is an isomorphism of real vector spaces.

Smoothness of the vector space structure maps. Very little additional work is needed to verify that the zero map and scalar multiplication are smooth. For the zero map we simply note that $``H_\alpha^{-1}\circ z\circ h_\alpha"$ is simply the smooth map $z_\alpha$, and for scalar multiplication we note that $``H_\alpha^{-1}\circ\mu\circ(\mathbf{1_R}\times H_\alpha)"$ is simply the smooth map μ_α. However, some extra work is needed to show that the addition map Σ is smooth, and in particular we need to define a smooth structure on the domain of Σ. It will simplify the formulas to denote $D``h_\beta^{-1}h_\alpha"$ by $L_{\beta\alpha}$.

The smooth atlas for $T(M)\times_M T(M)$ can be constructed by taking the maps

$$Q_\alpha : U_\alpha\times\mathbf{R}^n\times\mathbf{R}^n\to T(M)\times_M T(M)$$

whose images are the subsets

$$\mathrm{Image}\,(H_\alpha\times H_\alpha)\cap(T(M)\times_M T(M))$$

of $T(M)\times T(M)$ and whose values are given by $``(H_\alpha\times H_\alpha)^{-1}\circ Q_\alpha"(x,\mathbf{v},\mathbf{w}) = (x,\mathbf{v};\ x,\mathbf{w})$. By the way, this shows that $T(M)\times_M T(M)$ is in fact a topological $3n$-manifold. We need to check that the transition maps are diffeomorphisms, but this drops out of the formula $``Q_\beta^{-1}Q_\alpha"(x,\mathbf{v},\mathbf{w}) = (``h_\beta^{-1}h_\alpha"(x), L_{\beta\alpha}(x)\mathbf{v}, L_{\beta\alpha}(x)\mathbf{w})$.

With the smooth atlas for $T(M)\times_M T(M)$ we can finish the verification of the smoothness of the addition map as before. For each α the map $``H_\alpha^{-1}\circ\Sigma\circ Q_\alpha"$ corresponds to the map σ_α given by

$$\sigma_\alpha(u,\mathbf{v},\mathbf{w}) = (u,\mathbf{v}+\mathbf{w})$$

which is a smooth mapping.

5. Functoriality of the tangent construction

The aim of this section is to show that *the tangent space construction for smooth manifolds extends also yields a compatible construction for smooth maps of smooth manifolds.* Here is a summary of the main construction:

THEOREM. *Let $f : M\to N$ be a smooth map of smooth manifolds (we suppress the atlases here to simplify the notation) where $\dim M = m$ and $\dim N = n$. Then there is a unique smooth map $T(f) : T(M)\to T(N)$ such that the following hold:*
(i) For each $p\in M$, $T(f)$ sends $T_p(M)$ linearly to $T_{f(p)}(N)$.
(ii) If we have smooth charts (U_α, h_α) for M and (V_β, k_β) for N such that $f(h_\alpha(U_\alpha))\subset k_\beta(V_\beta)$ and the maps for the associated charts in the tangent space atlases are denoted by H_α and K_β, then $T(f)$ maps $H_\alpha(U_\alpha\times\mathbf{R}^m)$ into $K_\beta(V_\beta\times\mathbf{R}^n)$ and $``K_\beta^{-1}\circ T(f)\circ H_\alpha"(x,\mathbf{v})$ is equal to $(``k_\beta^{-1}\circ f\circ h_\alpha"(x),\ D``k_\beta^{-1}\circ f\circ h_\alpha"(x)\mathbf{v})$.

The formula in part (ii) shows how to define the map, and it is another elementary but tedious exercise in bookkeeping to show that this is well-defined.

In the language of category theory, the next result states that the constructions $M\rightsquigarrow T(M)$ and $f\rightsquigarrow T(f)$ define a *covariant functor* from the category of smooth manifolds to itself.

Theorem. *The construction $f \rightsquigarrow T(f)$ has the following properties:*
(*a*) $T(\mathbf{1}_M) = \mathbf{1}_{T(M)}$.
(*b*) *If $f : M \to N$ and $g : N \to P$ are smooth then $T(g \circ f) = T(g) \circ T(f)$.*

Both of these follow directly from the description of the associated map of tangent spaces. In the first case, this is true because the derivative of an identity map is always an identity matrix, and in the second case this is true by the Chain Rule.

Given a smooth map $f : M \to N$ and $p \in M$ it is often convenient to use $T_p(f)$ to denote the associated linear map from $T_p(M)$ to $T_{f(p)}(N)$.

5. Immersions, submersions and embeddings

We have already discussed the concepts in the title for smooth maps of open subsets in Euclidean spaces. The machinery developed thus far allows us to discuss these concepts also for arbitrary smooth manifolds.

Definition. If $f : M \to N$ is a smooth map of smooth manifolds (once again suppressing the atlases for notational simplicity), then f is said to be a **submersion** if for each $p \in M$ the linear map $T_p(f)$ is onto.

Definition. If $f : M \to N$ is a smooth map of smooth manifolds, then f is said to be an **immersion** if for each $p \in M$ the linear map $T_p(f)$ is 1–1.

Definition. If $f : M \to N$ is a smooth map of smooth manifolds, then f is said to be an **embedding** if it is a 1–1 immersion and maps M homeomorphically to $f(M)$.

Local characterizations for submersions and immersions were previously given when M and N are open subsets of Euclidean spaces, and it follows immediately that these also hold if M and N are arbitrary smooth manifolds.

Example. Not every 1–1 immersion is an embedding. Consider the figure 8 curve $\varphi(t) = (\sin 2t, \sin t)$ for $t \in (0, 2\pi)$. The image of this curve is a figure 8 where the crossing point is the origin, and therefore the image is not a manifold.

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On the other hand, it is an elementary exercise to check that $\varphi'(t)$ is never zero and that φ is 1–1 on the open interval $(0, 2\pi)$. Of course, one can extend the definition of φ to all real values of t but then the function will not be 1–1.

Submanifolds

The preceding definitions provide the tools we need to formulate a definition of submanifold that includes open subsets of smooth manifolds and level sets of regular values.

Definition. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds such that N is a subset of M. Then $(N, \mathcal{B})$ is said to be a *smooth submanifold* of $(M, \mathcal{A})$ if the inclusion map $i : N \subset M$ is smooth.

An important relation between embeddings and submanifolds is contained in the following result:

Proposition. *Let $F : (N, \mathcal{B}) \to (M, \mathcal{A})$ be a smooth embedding. Then there is a smooth atlas $\mathcal{C}$ on $f(N)$ such that*

(i) if $i : f(N) \to M$ is the inclusion map, then i is the inclusion of $f(N)$ as a smooth submanifold,

(ii) if $g : N \to f(N)$ is the map of sets such that $f = i \circ g$, then g is a diffeomorphism.

Proof. By hypothesis the map f defines a homeomorphism onto its image, and therefore the map g is a homeomorphism. As in the text, we can find a smooth atlas $\mathcal{C}$ on $f(N)$ such that g defines a diffeomorphism from $(N, \mathcal{B})$ to $(f(N), \mathcal{C})$; specifically, the charts in $\mathcal{C}$ all have the form $(U_\alpha, g \circ h_\alpha)$, where (U_α, h_α) is a smooth chart in $\mathcal{B}$. Since $i = f \circ g^{-1}$, it follows that i is also a smooth immersion.

The following result is crucial in the study of submanifolds.

Proposition. *Suppose that $(N, \mathcal{B})$ is a smooth submanifold of $(M, \mathcal{A})$ where $n = \dim N$ and $m = \dim M$. Then for each $p \in N$ there exist smooth charts (U, h) and $(U \times (-\delta, \delta)^{m-n}, k)$ for N and M respectively such that $p = h(x) = k(x, 0)$ for some $x \in U$, the inclusion map is given locally by "$k^{-1} \circ i \circ h$"$(u) = (u, 0)$, and $h(U) = k(U \times (-\delta, \delta)^{m-n}) \cap N$.*

Proof. Since the inclusion mapping i is an immersion it follows that there are smooth charts (U_0, h_0) and $(U_0 \times (-\delta_0, \delta_0)^{m-n}, k_0)$ satisfying all the conditions except perhaps the final one. Without further consideration it is conceivable that the image of k_0 contains other points of N, and we need shrink U_0 and δ_0 to remove any such points.

Since $h(U_0)$ is open in M there is an open set $W \subset M$ such that $W \cap M = h(U_0)$. Consider the intersection $W \cap k_0(U_0 \times (-\delta_0, \delta_0)^{m-n})$. One can then find an open set $U_0 \subset U$ containing x and some $\delta > 0$ satisfying $\delta_0 < \delta$ such that

$$k_0(U \times (-\delta, \delta)^{m-n}) \subset W \cap k_0(U_0 \times (-\delta_0, \delta_0)^{m-n}).$$

The entire conclusion of the proposition now holds for $k = k_0 | U \times (-\delta, \delta)^{m-n}$.

There are complementary results for recognizing smooth submanifolds of a smooth manifold. For example, the main result on level sets of regular values of smooth functions extends to smooth manifolds:

Proposition. *Let $f : M \to N$ be a map of smooth manifolds and let $p \in N$ be a nontrivial regular value; i.e., $p = f(x)$ for some $x \in M$ and if $f(y) = p$ then $T_y(f)$ maps $T_y(M)$ onto $T_p(N)$. Then $V = f^{-1}(\{p\})$ is a smooth $(m - n)$-dimensional smooth submanifold of N.*

The proof of this result similar to the local proof and is left to the reader as an exercise. The following result may be proved by the same method.

Proposition. *Let $(M, \mathcal{A})$ be a smooth m-manifold, and let $N \subset M$ be a topological n-manifold such that for each $p \in M$ there exists a smooth chart of the form $(U \times (-\delta, \delta)^{m-n}, k)$ for $\delta > 0$ and open set U in $\mathbf{R}^n$ such that $p = k(x, 0)$ for some $x \in U$ and $N \cap k(U \times (-\delta, \delta)^{m-n}) = k(U \times 0)$. Then there is a smooth atlas $\mathcal{B}$ for N such that $(N, \mathcal{B})$ is a smooth submanifold of $(M, \mathcal{A})$.*

In fact, a smooth atlas for N is given by charts of the form $(U \times \{0\}, k | U \times \{0\})$.

Here is another result that generalizes an earlier statement in the note, *Level sets of regular values – II*.

Theorem. *Suppose that N is a smooth submanifold of M, let $i : N \subset M$ be the inclusion map, let P be a smooth manifold, and let $g : P \to N$ be a set-theoretic map. Then g is continuous if and only if $i \circ g$ is continuous, and g is smooth if and only if $i \circ g$ is smooth.*

Once again the proof is similar to the earlier special case.

Finally, one can use smooth partitions of unity to prove the following result; the argument is similar to the proof of the corresponding result for topological manifolds.

Theorem. *If M is a compact smooth manifold, then there is a smooth embedding of M into some Euclidean space.*

6. Vector bundles

Tangent spaces are a basic example of a central class of objects in the topology of smooth manifolds. There are many instiances where one needs to consider a smoothly parametrized family of k-dimensional vector spaces over an n-dimensional manifold. For the tangent bundle we have $k = n$, but in many other contexts we want k and n to be independent of each other.

Example. There is a canonical 1-dimensional vector bundle defined on real projective space $M = \mathbf{RP}^n$ as follows: Let E be the set of all points $(x, y) \in \mathbf{RP}^n \times \mathbf{R}^{n+1}$ such that either $y = 0$ or y is a set of homogeneous coordinates for x. If we view projective space as the quotient of S^n by the equivalence relation $v \sim v' \Leftrightarrow v = \pm v'$, then the condition on y means that y is a scalar multiple of x.

There is an obvious continuous map π from E to M defined by projection onto the first factor, and it turns out that one can define vector space operations $z : M \to E$ (the zero map), $\mu : \mathbf{R} \times E \to E$ (scalar multiplication), and $\Sigma : E \times_M E \to E$ (vector addition) that make each inverse image $\pi^{-1}(\{x\})$ into a 1-dimensional real vector space. Furthermore, each point x has an open neighborhood U such that $\pi^{-1}(U) \cong U \times \mathbf{R}$, and in fact one can choose U such that there is a homeomorphism $h : U \times \mathbf{R} \to \pi^{-1}(U)$ such that $\pi(h(x,t)) = x$ and for each $x \in U$ the map h determines a vector space isomorphism from $\{x\} \times \mathbf{R}$ to $\pi^{-1}(\{x\})$. None of this is particularly difficult to verify, but it is time consuming and it would distract us from our objective, so we shall not give the proofs here.

Remark. The space E is homeomorphic to the quotient of $S^n \times \mathbf{R}$ by the equivalence relation $(x,t) \sim (y,s) \Leftrightarrow (y,s) = \pm(x,t)$. This follows by consideration of the map

$$\widetilde{\Phi} : S^n \times \mathbf{R} \to \mathbf{RP}^n \times \mathbf{R}^{n+1}$$

sending (x,t) to $([x], tx)$ where $[x]$ denotes the equivalence class in $\mathbf{RP}^n$ represented by x. This map sends both points in an equivalence class for $\sim$ to the same point in the codomain and therefore passes to a continuous map Φ on the quotient space, and E is equal to the image of Φ; one can also show that Φ maps the quotient space homeomorphically onto E because it is a closed map (in fact, both $\widetilde{\Phi}$ and Φ are *proper*).

There is a clear analogy between the vector space structure described above and that of a tangent bundle, and we would like to have general methods for working with such objects.

The first step is to formulate the topological theory.

Definition. Let $\pi : E \to B$ be a continuous map. We shall say that π is a *topological fiber bundle projection* if for each $x \in B$ there is an open neighborhood V of x such that $\pi^{-1}(V)$ is homeomorphic to a product space $V \times F$ by a homeomorphism $h : V \times F \to \pi^{-1}(V)$ such that $\pi(h(v, z) = v$ for all (v, z). If for all $y \in B$ the spaces $\pi^{-1}\{y\})$ are all homeomorphic to some fixed space F we shall say that F is the *fiber* of π. Frequently E is called the *total space* and B is called the *base space*, and we often simply say that $\pi : E \to B$ is a *fiber bundle*..

Examples.
1. Take $E = F \times B$ with π given by projection onto B. This is called a *trivial* fiber bundle.
2. If π is a covering space projection, then it is a fiber bundle projection, and if E and B are "nice" spaces (Hausdorff, connected, locally path connected, semilocally simply connected) then π has a discrete space as its fiber. — In particular, this provides a host of examples of fiber bundles that are not trivial bundles; for example, take the simply connected covering of a topological manifold with a nontrivial fundamental group. This can be seen by noting that the total space of a trivial fiber bundle with discrete fiber containing at least two points is disconnected while the simply connected covering space has such a fiber but is connected.
3. The open Möbius strip X can be viewed as a nontrivial fiber bundle with fiber given by the open interval $(-1, 1)$. If we view this space in the traditional way as, say, the quotient space of $[-10, 10] \times (-1, 1)$ by the equivalence relation whose equivalence classes are given by the two point sets

$$\{(10, t),\ (-10, -t)\} \qquad (t \in (-1, 1))$$

and the one point sets containing the points (s, t) for $s \in (-10, 10)$, then we have a canonical continuous map $q : X \to S^1$ sending the class of (u, t) to $\exp(\pi i u/10)$. For all points in S^1 except -1 one can take the special neighborhood $V \subset S^1$ to be the complement of -1, and for -1 one can in fact take V to be the complement of $+1$. — The nontriviality of this fiber bundle projection can be seen as follows: Consider the simple closed curve $C \subset X$ given by sending $u \in [-10, 10]$ to the class of $(u, 0) \in X$. It is well known that the complement $X - C$ is connected (this is essentially what one gets if one cuts a Möbius strip along the center curve). On the other hand, the Jordan Curve Theorem implies that the complement of a simple closed curve in $S^1 \times (-1, 1)$ has two components.
4. The tangent bundle to a smooth n-manifold is a fiber bundle whose fiber is $\mathbf{R}^n$. Similarly, the projection $T(M) \times_M T(M)$ is a fiber bundle whose fiber is $\mathbf{R}^n \times \mathbf{R}^n$. — A natural and fundamental question in the theory of manifolds is to *determine whether the tangent bundle to a given n-manifold is a trivial bundle in the sense described above.*
5. The mapping torus construction for a homeomorphism $h : M \to M$, described in the note on atlases ($q.v.$), gives an example of a fiber bundle over the circle whose fiber is the original manifold M. It is a trivial bundle if and only if the homeomorphism satisfies a technical condition (pseudo-isotopic to the identity). The Klein bottle is an example of such a construction such that the corresponding fiber bundle is not trivial.

A k-dimensional vector bundle over a topological space B is basically a fiber bundle with fiber $\mathbf{R}^k$ such that there is a vector space structure on each $\pi^{-1}(\{b\})$ that has the same sort of continuity properties one has for the tangent bundle. Specifically, one wants the zero section $z : B \to E$, the scalar multiplication $\mu : \mathbf{R} \times E \to E$ and the vector addition $\Sigma : E \times_B E \to E$ to be continuous maps. We shall also assume one more condition; namely, for each point $b \in B$ there is an open neighborhood V of b and a homeomorphism $h : V \times \mathbf{R}^k \to \pi^{-1}(V)$ such that $h(\pi(v, w)) = v$ and the map $h_b : \{b\} \times \mathbf{R}^k \to \pi^{-1}(\{b\}) := E_b$ is an isomorphism of vector spaces.

We say that $(\pi : E \to B, z, \mu, \Sigma)$ is a *trivial vector bundle* if there is a homeomorphism $H : E \to B \times \mathbf{R}^k$ such that for each $b \in B$ the map H sends E_b to $\{b\} \times \mathbf{R}^k$ by a linear isomorphism. — The question about tangent bundles can then be reformulated to determine whether the tangent bundle to an n-manifold is a trivial n-dimensional vector bundle.

It turns out that the tangent bundles for S^1 and S^3 are trivial but the tangent bundle to S^2 is not; these will be shown in subsequent material. A result of R. Bott and J. Milnor from the late nineteen fifties states that the only spheres with trivial tangent bundles are S^1, S^3 and S^7. In each of these cases the triviality of the tangent bundle is related to a classical finite dimensional algebra over the reals; the complex numbers, quaternions and the Cayley numbers respectively.

Suppose now that B is the underlying space of some smooth manifold, and let $\mathcal{B}$ be a smooth atlas for B. Given a k-dimensional vector bundle $\pi : E \to B$ it follows immediately that E is a topological $(n+k)$-manifold. It turns out that one can always impose a canonical smooth structure on E so that the projection map π is a smooth submersion, but for our purposes it will be enough to describe a more elementary way of imposing a canonical smooth structure on E.

Definition. In the setting above, a smooth atlas $\mathcal{E}$ for E is said to be a smooth vector bundle atlas over $\mathcal{B}$ if the following hold:

(i) For each smooth chart (U_α, h_α) in $\mathcal{B}$ there is a unique chart of the form $(U_\alpha \times \mathbf{R}^n, H_\alpha)$ in $\mathcal{E}$ such that $H_\alpha \circ \pi(x, \mathbf{v}) = h_\alpha(x)$.

(ii) For each α and $x \in U_\alpha$, the map H_α sends $\{x\} \times \mathbf{R}^k$ to $\pi^{-1}(h_\alpha(x))$ by a linear isomorphism.

If we take the smooth structure on E determined by (the maximal atlas associated to) $\mathcal{E}$ then it follows immediately that the bundle projection map π, the zero map and the scalar multiplication map are all smooth. As in the case of the tangent bundle, we need to put a smooth structure on $E \times_B E$ in order to discuss the smoothness of Σ; however, this can be done exactly as in the case of the tangent bundle.

At some point one needs to verify that equivalent atlases on B yield equivalent atlases on E. This is not particularly difficult but it is time-consuming and the arguments are not particularly enlightening, so we shall omit the details.

Vector bundle amalgamation data

In practice vector bundles are often constructed from the sort of data yielding the tangent bundle, so we shall describe the construction in a general form.

Let X be a topological space, let $\mathcal{U} = \{U_\alpha\}$ be an open covering of X and let $\{\psi_{\beta\alpha}\}$ denote the associated transition data; *i.e.*, $\psi_{\beta\alpha}$ is the homeomorphism identifying $V_{\beta\alpha} \cong U_\alpha \cap U_\beta \subset U_\alpha$ with $V_{\alpha\beta} \cong U_\beta \cap U_\alpha \subset U_\beta$. A k-dimensional *vector bundle preatlas over* $\mathcal{U}$ is a set of topological amalgamation data $(\{Y_\alpha\}, \{\Phi_{\beta\alpha}\})$ such that

(i) $Y_\alpha = U_\alpha \times \mathbf{R}^k$,

(ii) $\varphi_{\beta\alpha}$ maps $V_{\beta\alpha} \times \mathbf{R}^k$ to $V_{\alpha\beta} \times \mathbf{R}^k$ such that

$$\Phi_{\beta\alpha}(x, \mathbf{y}) = (\psi_{\beta\alpha}(x), F_{\beta\alpha}(x, \mathbf{y}))$$

where $F_{\beta\alpha}$ is continuous and each slice map $F_x : \{x\} \times \mathbf{R}^k \to \{\psi_{\beta\alpha}(x)\} \times \mathbf{R}^k$ is a vector space isomorphism.

The conditions describing a vector bundle preatlas are parallel to those on the amalgamation data for the tangent bundle, and in fact the methods used to construct and establish properties of the tangent bundle also allow one to construct a vector bundle over X from a vector bundle preatlas.

Realization Theorem. *Given a k-dimensional vector bundle preatlas as above, there is a k-dimensional vector bundle $\pi : E \to X$ such that*

(i) for every open set U_α in the open covering $\mathcal{U}$, there is a homeomorphism $H_\alpha : U_\alpha \times \mathbf{R}^k \to \pi^{-1}(U_\alpha)$ such that $\pi(H_\alpha(x\mathbf{y}) = x$ for all x,

(ii) for each pair of open sets U_α and U_β the transition map "$H_\beta^{-1} H_\alpha$" is equal to $\Phi_{\beta\alpha}$.

The space E is Hausdorff if X is Hausdorff, and E is second countable if X is second countable.

Alternate description. In the construction of the tangent bundle the counterparts of the maps $F_{\beta\alpha}$ have the form

$$F_{\beta\alpha}(x, \mathbf{y}) = [L_{\beta\alpha}(x)]\mathbf{y}$$

for continuous (in fact, smooth) maps from $V_{\beta\alpha}$ to the group $GL(k, \mathbf{R})$ of all invertible $k \times k$ matrices (recall that this group is in fact an open subset of the k^2-dimensional Euclidean space of all $k \times k$ matrices, and the matrix multiplication and inverse maps are smooth). The basic compatibility conditions for amalgamation data followed from identities of the form

(i) $L_{\alpha\beta}(x) = [L_{\beta\alpha}(\psi_{\alpha\beta}(x))]^{-1}$,
(ii) $L_{\gamma\alpha}(x) = L_{\gamma\beta}(\psi_{\beta\alpha}(x)) \cdot L_{\beta\alpha}(x)$.

A family of maps satisfying these identities is essentially a $GL(k)$-cocycle in the notation of Conlon, Section 3.4.

The construction method for the tangent bundle shows that a $GL(k)$-cocycle in the preceding sense yields a vector bundle, for using these maps one can form an associated preatlas with transition maps

$$\Phi(x, \mathbf{y}) = (\psi_{\beta\alpha}(x), L_{\beta\alpha}(x)\mathbf{y}).$$

In fact, given a vector bundle preatlas one can retrieve a $GL(k)$-cocycle from the following result:

Coadjunction Property. *Let W be a topological space, and let $\mathbf{F} : W \times \mathbf{R}^k \to \mathbf{R}^k$ be a continuous map such that for each $p \in W$ the restriction*

$$F|\{p\} \times \mathbf{R}^k : \{p\} \times \mathbf{R}^k \to \mathbf{R}^k$$

is a linear isomorphism. Then there is a unique continuous map $L : W \to GL(k, \mathbf{R})$ such that $F(x, \mathbf{y}) = [L(x)]\mathbf{y}$ for all $(x, \mathbf{y})$. Furthermore, if W is open in some Euclidean space and F is smooth, then L is also smooth.

Proof. Let $\{\mathbf{e}_1, \cdots, \mathbf{e}_k\}$ be the standard basis of unit vectors for $\mathbf{R}^k$ and let $\langle\ ,\ \rangle$ denote the usual inner product. If for each i, j we define

$$a_{i,j}(x) = \langle F(x, \mathbf{e}_j), \mathbf{e}_i\rangle.$$

These functions are continuous by the continuity of F and the inner product map

$$\langle\ ,\ \rangle : \mathbf{R}^k \times \mathbf{R}^k \to \mathbf{R}^k$$

and hence one can define a continuous function L such that $L(x)$ is the $k \times k$ matrix whose entries are given by the functions $a_{i,j}(x)$, it then follows that $F(x, \mathbf{y}) = [L(x)]\mathbf{y}$ for all $(x, \mathbf{y})$. Since each $F|\{x\} \times \mathbf{R}^k$ is a linear isomorphism it follows that $L(x)$ always lies in the open set $GL(k, \mathbf{R})$. The final statement regarding smoothness follows because the inner product is smooth.

Of course, there is a corresponding notion of *smooth vector bundle preatlas*, but a few changes are needed in order to formulate this. Given a smooth manifold M and a smooth atlas $\mathcal{A} = \{(U_\alpha, h_\alpha)\}$ for M, a k-dimensional smooth vector bundle atlas over $\mathcal{A}$ is a set of amalgamation data $(\{Y_\alpha\}, \{\Phi_{\beta\alpha}\})$ such that

(i) $Y_\alpha = U_\alpha \times \mathbf{R}^k$,

(ii) $\varphi_{\beta\alpha}$ maps $h_\alpha^{-1}(h_\beta(U_\beta)) \times \mathbf{R}^k$ to $h_\beta^{-1}(h_\alpha(U_\alpha)) \times \mathbf{R}^k$ by a diffeomorphism such that $\Phi_{\beta\alpha}(x, \mathbf{y})$ is equal to $(\text{“}h_\beta^{-1} h_\alpha\text{”}(x), F_{\beta\alpha}(x, \mathbf{y}))$, where $F_{\beta\alpha}$ is smooth and each slice map $F_x : \{x\} \times \mathbf{R}^k to \{\psi_{\beta\alpha}(x)\} \times \mathbf{R}^k$ is a vector space isomorphism.

Once again the point of the definition is that the structure leads to a vector bundle.

Smooth Realization Theorem. *Given a k-dimensional smooth vector bundle preatlas as above, there is a smooth k-dimensional vector bundle $\pi : E \to X$ such that*

(i) *for every smooth chart (U_α, h_α) in the atlas $\mathcal{U}$, there is a homeomorphism $H_\alpha : U_\alpha \times \mathbf{R}^k \to \pi^{-1}(h_\alpha(U_\alpha))$ such that $\pi(H_\alpha(x, \mathbf{y}) = h_\alpha(x)$ for all x,*

(ii) *for each pair of open sets U_α and U_β the transition map “$H_\beta^{-1} H_\alpha$” is equal to $\Phi_{\beta\alpha}$.*

Warning. For the tangent bundle of a smooth manifold M, given a smooth atlas $\mathcal{A}$ one can construct a smooth vector bundle preatlas over $\mathcal{A}$. This is usually NOT the case for arbitrary smooth vector bundles.

Constructions on vector bundles

Not surprisingly, there is a variety of techniques for constructing new vector bundles out of old ones.

Pullbacks. Given a fiber bundle $\pi : E \to B$ with fiber F and a subspace $Y \subset B$, the map

$$i^*\pi = \pi|\pi^{-1}(Y) : \ \pi^{-1}(Y) \to Y$$

is also a topological fiber bundle with fiber F. It is called the *induced bundle* or the *pullback* of π with respect to the inclusion map $i : Y \subset B$. It is also called the *restriction* of the bundle to Y. If the fiber bundle has a k-dimensional vector bundle structure then the restricted bundle inherits a k-dimensional vector bundle structure, and the inclusion $\pi^{-1}(Y) \to E$ sends each k-dimensional vector space in the domain isomorphically to a vector space of the same type in the codomain.

This construction can be extended to an arbitrary continuous map. Here is a fast way of doing so. Given a continuous map $f : X \to Y$ we can write f as a composite $p_Y \circ \Gamma(f)$, where p_Y is the projection $X \times Y \to Y$ and $\Gamma(f) : X \to Y$ is the graph of f ($\Gamma(f)(x) = (x, f(x))$). Given a fiber bundle $\pi : E \to Y$ the pullback

$$f^*\pi : f^*E \to X$$

is determined by the the inverse image of the graph of f with respect to the map

$$\mathbf{1}_X \times \pi : X \times E \to X \times Y.$$

If π has a k-dimensional vector bundle structure then the pullback has an induced structure, and there is a map $\widetilde{f} : f^*\pi \to E$ such that for each $x \in X$ the map $\widetilde{f}$ sends the k-dimensional vector space over x isomorphically to the corresponding vector space over $f(x)$.

The following elementary observation reflects an important use of the pullback construction:

Proposition. *If* $(\pi,$ *etc.*$)$ *is a trivial vector bundle and* $i : Y \subset B$ *is inclusion, then* $(i^*\pi,$ *etc.*$)$ *is also trivial.*

In particular, using this one can reduce the proof of nontriviality of the canonical line bundle over $\mathbf{RP}^n$ to the case $n = 1$ In this case $i^*\pi$ turns out to be equivalent to the projection from the Möbius strip to the circle that was described above.

Exercise. Show that the pullback of a trivial (vector) bundle is a trivial (vector) bundle.

One of the most basic constructions in linear algebra is the direct sum. For vector bundles over the same space B there is a corresponding notion of direct sum:

Definition. Let $\pi : E \to B$ and $\rho : E' \to B$ be the bundle projections for m- and n-dimensional vector bundles over the same base space B. Let $E \times_B E'$ be the set of all $(a, b) \in E \times E'$ such that $\pi(a) = \rho(b)$, and define $\sigma : E \times_B E' \to B$ send (a, b) to $\pi(a) = \rho(b)$.

Proposition. *There is an* $(m + n)$*-dimensional vector bundle structure on* $(\sigma,$ *etc.*$)$ *such that the following hold:*

(i) *There are continuous maps* $J : E \to E \times_B E'$ *and* $J' : E' \to E \times_B E'$ *such that* J *and* J' *send the inverse image of each point* x *to the inverse image of* x *in* $E \times_B E'$.

(ii) *For each* $x \in B$ *the maps of fibers*

$$J_x : \pi^{-1}(\{x\}) \to \sigma^{-1}(\{x\}), \qquad J'_x : \rho^{-1}(\{x\}) \to \sigma^{-1}(\{x\})$$

are linear and determine a direct sum decomposition of $\sigma^{-1}(\{x\})$.

One frequently uses the notation $\sigma \cong \pi \oplus \rho$ to summarize this situation.

Proof. Given a point $x \in B$ there are open neighborhoods U and V of x such that the restriction of the vector bundles to U and V are equivalent to product bundles $U \times \mathbf{R}^m \to U$ and $V \times \mathbf{R}^n \to V$. If $W = U \cap V$ then the inverse image of W in $E \times_B E'$ is homeomorphic to $W \times \mathbf{R}^m \times \mathbf{R}^n$ such that σ corresponds to projection onto the first coordinate. Furthermore, under this homeomorphism the vector space structure on each fiber $\sigma^{-1}(\{y\})$ can be checked to be the standard vector space structure on the direct sum

$$\mathbf{R}^m \oplus \mathbf{R}^n \cong \mathbf{R}^m \times \mathbf{R}^n$$

and over W the maps J and J' correspond to the injections

$$W \times \mathbf{R}^m \cong W \times (\mathbf{R}^m \oplus \mathbf{0}) \to W \times (\mathbf{R}^m \oplus \mathbf{R}^n)$$

and

$$W \times \mathbf{R}^n \cong W \times (\mathbf{0} \oplus \mathbf{R}^n) \to W \times (\mathbf{R}^m \oplus \mathbf{R}^n)$$

respectively.

Inner products are another basic concept in linear algebra that extend to vector bundles.

Definition. Let $\Pi = (\pi, \ etc.)$ be a vector bundle; for each $x \in B$ denote the vector space $\pi^{-1}(\{x\})$ by E_x and denote the canonical projection $E \times_B E \to E$ by ${}_2\pi$. A (continuous) *riemannian metric* on Π is a continuous map

$$g : E \times_B E \to \mathbf{R}$$

such that for each $x \in B$ the map g defines an inner product

$$g_x : E_x \times E_x \cong {}_2\pi^{-1}(\{x\}) \to \mathbf{R}$$

If we are given a (maximal) smooth atlas on B and an associated richer structure of a smooth vector bundle for Π, the metric will be said to be *smooth* if g is smooth.

Of course inner products play an important role in the study of real vector spaces, and riemannian metrics play at least an equally important role in the study of smooth manifolds (and they are arguably even more important for the latter).

The following elementary observation is important for our purposes.

Proposition. *Let V be a real vector space, let $g_i : V \times V \to \mathbf{R}$ be an inner product on V for $i = 1$ or 2, and let t be a real number in the closed interval $[0, 1]$. Then $tg_1 + (1 - t)g_2$ is also an inner product on V.*

This proposition has an extremely far-reaching consequence: **Theorem.** *Let Π be a vector bundle such that the base space B is paracompact Hausdorff (e.g., take B to be a second countable topological manifold). Then there is a riemannian metric on Π. If B has a smooth atlas and we are given an associated richer structure of a smooth vector bundle for Π, then a smooth riemannian metric exists.*

Proof. We begin with the topological conclusion. Take a locally finite open covering $\mathcal{U}$ of$B$ such that the bundle looks like a product over each open set U_α in the open covering. For each U_α a riemannian metric f_α can be constructed on $\pi^{-1}(U_\alpha) \cong U_\alpha \times \mathbf{R}^k$ simply by taking the standard inner product on each vertical slice $\{u\} \times \mathbf{R}^k$. Let φ_α be a partition of unity subordinate to $\mathcal{U}$. Then we can extend each product function $\varphi_\alpha \cdot f_\alpha$ to a continuous map g_α from $E \times_B E$ to $\mathbf{R}$, and if $W_\alpha \subset B$ is the set of points where $\varphi_\alpha \neq 0$, then then the restriction of g_α to ${}_2\pi^{-1}(W_\alpha)$ is a riemannian metric. By local finiteness of the open covering the sum $g = \Sigma_\alpha g_\alpha$ is meaningful; but for each $x \in B$ there is an α such that $\varphi_\alpha(x) \neq 0$, and therefore by the proposition it follows that g defines a riemannian metric on all of $E \times_B E$.

The existence of a smooth metric follows by taking a smooth atlas $\mathcal{A} = \{(V_\beta, k_\beta)\}$ for B such that each image $k_\beta(V_\beta)$ lies inside some U_α from $\mathcal{U}$ and the resulting open covering $\mathcal{V} = \{k_\beta(V_\beta)\}$ is locally finite (verify that one can find an atlas so that both conditions hold). One can then choose a *smooth* partition of unity φ_b subordinate to $\mathcal{V}$, and the corresponding metrics f_β over the open sets in this open covering are smooth by construction. Therefore the sum $g = \Sigma_\beta \varphi_\beta \cdot f_\beta$ is a smooth riemannian metric.

To illustrate the usefulness of riemannian metrics we shall use them to define the length of a smooth curve in a smooth manifold.

Notation. If g is a smooth riemannian metric on the tangent bundle τ_M and $\Gamma : (c, d) \to M$ is a smooth curve and $t \in (c, d)$, then $\Gamma'(t) \in T_{\Gamma(t)}(M)$ is the image of $(t, 1)$ under the canonical identification

$$(c, d) \times \mathbf{R} \cong T\left((c, d)\right)$$

followed by the associated map of tangent spaces

$$T(\Gamma) : T\left((c,d)\right) \to T(M).$$

If $c < a < b < d$ then the **length** of $\Gamma|[a,b]$ is given by the integral

$$\int_a^b \sqrt{g\left(\Gamma'(t), \Gamma'(t)\right)}\, dt$$

which exists by the smoothness of g and Γ.

One can also define the length of a smooth curve if it is only defined on the closed interval $[a,b]$, but this will be left to the reader in order to keep the discussion relatively brief.

Many important examples of riemannian metrics arise from inclusions of submanifolds. For example, suppose that M is a smooth submanifold of $\mathbf{R}^n$ and i denotes the inclusion map. Then $T(i)$ defines a smooth embedding of $T(M)$ in $T(\mathbf{R}^n) \cong \mathbf{R}^n \times \mathbf{R}^n$, and if P_2 denotes projection onto the coordinate then for each $p \in M$ the composite $P_2 \circ T(i)|T_p(M)$ is a 1–1 linear transformation. Therefore if $\mathbf{v}$ and $\mathbf{w}$ lie in $T_p(M)$ then

$$\langle P_2 \circ T(i)\mathbf{v}, P_2 \circ T(i)\mathbf{w}\rangle$$

defines a riemannian metric on $T(M)$. In the classical theory of surfaces where $n = 3$ and $\dim M = 2$, this riemannian metric is called the **First Fundamental Form** of the embedded surface.

In mathematical physics (or older books on tensor analysis and differential geometry) a riemannian metric is often described locally over each chart in an atlas, with a compatibility requirement for the definitions over different charts. We shall state a version of this approach from the perspective of our notes, but first we need an observation and some notation. If $\mathbf{Symm}(n)$ is the set of all symmetric $n \times n$ matrices, then $\mathbf{Symm}(n)$ is a subspace of the space of all $n \times n$ matrices, and its dimension is $\frac{n^2+n}{2}$. Given a matrix A we shall denote its transpose by ${}^{\mathbf{T}}A$.

Proposition. *Let M^n be a smooth manifold, let $\mathcal{A} = (U_\alpha, h_\alpha)$ be a smooth atlas for M, and let $(U \times \mathbf{R}^n), H_\alpha)$ be the chart for the tangent space $T(M)$ associated to (U_α, h_α). Suppose that for each α we have a smooth map $G_\alpha : U_\alpha \to \mathbf{Symm}(n)$ such that for all $x \in U_\alpha$ and nonzero $\mathbf{v} \in \mathbf{R}^n$ we have ${}^{\mathbf{T}}\mathbf{v}G_\alpha(x)\mathbf{v} > 0$, and that for all α and β we have $G_\beta(\text{“}h_\beta^{-1}h_\alpha\text{”}(x) = {}^{\mathbf{T}}L_{\alpha\beta}(x)G_\alpha(x)L_{\alpha\beta}(x)$, where $L_{\alpha\beta}(x)$ is equal to $D\,\text{“}h_\beta^{-1}h_\alpha\text{”}(x)$. Then there is a smooth riemannian metric g on the tangent bundle of M such that*

$$g\left(H_\alpha(x,\ \mathbf{v}), H_\alpha(x,\ \mathbf{w})\right) = {}^{\mathbf{T}}\mathbf{w}G_\alpha(x)\mathbf{v}$$

for all x, $\mathbf{v}$ and $\mathbf{w}$.

Notation. If the entries of G_α are denoted by $g_{i,j}$ then the latter are smooth functions on U_α and the classical presentation of the metric locally is an expression of the form

$$\sum_{i,j} g_{i,j}(u)\, dx^i\, dx^j$$

(and frequently the summation sign is suppressed, the convention being that if a variable appears twice then one sums over it).

Footnote. A celebrated theorem proved by J. Nash in the nineteen fifties states that *every smooth riemannian metric can be realized as a metric coming from some smooth embedding of the manifold in a Euclidean space.* On the other hand, another celebrated theorem proved by D. Hilbert at the beginning of the twentieth century shows that one cannot realize the hyperbolic metric on the unit disk $\mathbf{H}^2$ in $\mathbf{R}^2$, which is defined by the formula

$$\frac{dx^2 + dy^2}{\left(1 - (x^2 + y^2)\right)^2}$$

as the First Fundamental Form for a smooth embedding of $\mathbf{H}^2$ as a closed subset of $\mathbf{R}^3$.

Warning. Inner products are special cases of *symmetric nondegenerate bilinear forms*, for which the positive definiteness condition

$$\mathbf{v} \neq 0 \qquad \Rightarrow \qquad \langle \mathbf{v}, \mathbf{v} \rangle \ > \ 0$$

is replaced by a nondegeneracy condition: *For each nonzero* $\mathbf{v}$ *there is a vector* $\mathbf{w}$ *such that* $\langle \mathbf{v},\ \mathbf{w} \rangle \neq 0$.

Such forms on a finite-dimensional real vector space V are classified up to equivalence by their *type*, which can be viewed as an ordered pair of nonnegative integers (r, s) such that $r + s = \dim V$ and there are subspaces V_+ and V_- such that

(*i*) we have $V_+ \cap V_- = \{\mathbf{0}\}$, $V_+ + V_- = V$, $\dim V_+ = r$ and $\dim V_- = s$,
(*ii*) if $\mathbf{x} \in V_+$ and $\mathbf{y} \in V_-$ then $\langle \mathbf{x},\ \mathbf{y} \rangle = 0$,
(*iii*) if $\mathbf{x} \in V_+$ is nonzero then $\langle \mathbf{x}, \mathbf{x} \rangle\ > 0$ and if $\mathbf{y} \in V_-$ is nonzero then $\langle \mathbf{y}, \mathbf{y} \rangle\ < 0$.

A riemannian metric is merely the special case where $s = 0$. We shall use the term *indefinite metric of type* (r, s) to denote a map g as above such that for each $z \in B$ the restriction to $E_z \times E_z$ is a nondegenerate symmetric bilinear form of type (r, s) where $s > 0$.

There are important contexts in which one wishes to consider analogs of riemannian metrics given by maps $g : T(M) \times_M T(M) \to \mathbf{R}$ such that the restriction to each $T_p(M) \times T_p(M)$ is a nondegenerate bilinear form that is not an inner product. For example, in relativity one considers *Lorentz metrics* on 4-dimensional manifolds for which the type is $(3, 1)$. Closely related objects are needed in classical mechanics, where considers maps $\Omega : T(M) \times_M T(M) \to \mathbf{R}$ that are bilinear, nondegenerate and *skew-symmetric* (*i.e.*,

$$\Omega\,(\mathbf{y}, \mathbf{x}) \ = \ -\Omega\,(\mathbf{x}, \mathbf{y})$$

for all $\mathbf{x}$ and $\mathbf{y}$); such a structure is called a *pre-symplectic structure* on M. It is not possible to construct indefinite metrics on arbitrary tangent bundles using partitions of unity as for riemannian metrics because the analog of the convexity proposition does not hold. In fact, indefinite metrics of a prescribed type and pre-symplectic structures do not necessarily exist on an arbitrary manifold.

The Second Fundamental Form

The **Second Fundamental Form** of an oriented hypersurface $M^{n-1} \subset \mathbf{R}^n$ is another example of a smooth map

$$T(M) \times_M T(M) \to \mathbf{R}$$

whose restriction to each $T_p(M) \times T_p(M) \cong {}_2\tau_M^{-1}(\{p\})$ is symmetric and bilinear. However, this map may be degenerate at some points (or even zero everywhere!). The orientation on an oriented hypersurface is essentially given by a unit normal vector field

$$\mathbf{N} : M \to \mathbf{R}^n$$

such that $\mathbf{N}(p)$ is perpendicular to $T_p(M)$, where the latter is viewed as a subspace of $\{p\} \times \mathbf{R}^n$ via the linear injection from $T_p(M)$ to $\{p\} \times \mathbf{R}^n$ induced by inclusion.

If $\mathbf{v} \in T_p(M)$ then it is not difficult to show that the image of $T_p(N)\mathbf{v}$ in $\{p\} \times \mathbf{R}^n$ is perpendicular to $\mathbf{N}(p)$ and therefore lies in $T_p(M)$. This means that $T(\mathbf{N})$ defines a smooth map S from $T(M)$ to itself such that for each $p \in M$ the map S send $T_p(M)$ to itself linearly. If we let $\mathbf{F}^{\mathbf{I}}_M$ denote the first fundamental form, then the Weingarten map has the self adjointness property

$$\mathbf{F}^{\mathbf{I}}_M \left(S(\mathbf{v}),\ \mathbf{w} \right) = \mathbf{F}^{\mathbf{I}}_M \left(\mathbf{v},\ S(\mathbf{w}) \right)$$

for all

$$(\mathbf{v}),\ \mathbf{w}) \in T(M) \times_M T(M)$$

and the second fundamental form is defined to be

$$\mathbf{F}^{\mathbf{II}}_M \left(\mathbf{v}),\ \mathbf{w} \right) = \mathbf{F}^{\mathbf{I}}_M \left(S(\mathbf{v}),\ \mathbf{w} \right).$$

If $n = 2$ the Weingarten map provides a very neat way to handle some classical concepts from the differential geometry of oriented surfaces in $\mathbf{R}^3$. Since S is self-adjoint, it has an orthonormal basis of (real) eigenvectors with real eigenvalues. The two eigenvalues of S at the point p are the *principal sectional curvatures* at p, half the trace of S at p is the *mean curvature* at p, and the determinant of S at p is the *Gaussian curvature* at p. An important result in differential geometry (the **Theorema Egregium** of Gauss) states that the Gaussian curvature only depends upon the FIRST Fundamental Form.