Constructing a cone from a sheet of paper

It is well known that we can construct the lateral surface of a right circular cone from a flat sheet of paper as follows:

1. Start by cutting out a disk of radius s from the sheet of paper.
2. Next, remove a circular sector of radian angle measure α from the disk, retaining the radii along the edges of that sector. Let W denote the remaining piece of the disk.
3. Finally, glue together the two radii in the second step.

If this is done, then have the lateral surface of a right circular cone (without the base). The *slant height* of the cone, which is the distance from a point on the circular base to the vertex of the cone, will be equal to s, and the radius t of the base will be $s\alpha/2\pi$. By the Pythagorean Theorem the altitude h of the cone will be $\sqrt{s^2 - t^2}$.

The purpose of this note is to justify the preceding construction mathematically. *Warning:* This argument requires input from multivariable calculus and, to a limited extent, the elementary differential geometry of surfaces.

The proof

The justification has the following steps:

A. Showing that the construction defines a continuous mapping σ from W to the cone in space defined by the equations and inequalities

$$x^2 + y^2 = \frac{s^2 t^2}{s^2 - t^2} z^2 , \qquad 0 \le z \le h .$$

This map is 1–1 except on the radii segments which it identifies.

B. Showing that σ is an isometric parametrization of the cone away from the vertex point. In other words, we want to verify that the First Fundamental Form $E\,dx\,dx + 2F\,dx\,dy + G\,dy\,dy$ satisfies $EG - F^2 = 1$.

Construction of the parametrization σ. This is best done using polar coordinates (r, θ) where $0 < r \le s$:

$$\sigma(r, \theta) = \left(\frac{tr}{s} \cos\left(\frac{s\theta}{t}\right), \frac{tr}{s} \sin\left(\frac{s\theta}{t}\right), r\sqrt{1 - \frac{t^2}{s^2}} \right)$$

The definitions imply that $\sigma(r, \theta) = \sigma(r', \theta')$ if and only if $r = 0$ or $r > 0$ and $\theta - \theta'$ is an integral multiple of $2\pi\, t$. Furthermore, it follows immediately that the image of σ contains all points defined by the equations and inequalities in (A.).

The proof of the second part reduces to verifying that the partial derivative vectors

$$\frac{\partial \sigma}{\partial x} , \qquad \frac{\partial \sigma}{\partial y}$$

form an orthonormal set. There will be five main steps in doing this:

(1) Computing the partial derivatives of σ with respect to r and θ using the formulas given above.

(2) Computing the dot products of these partial derivative vectors with each other and themselves.

(3) Expressing the partial derivatives of σ with respect to r and θ as linear combinations of the corresponding partial derivatives with respect to x and y.

(4) Using the preceding step to express the partial derivatives of σ with respect to x and y as linear combinations of the corresponding partial derivatives with respect to r and θ.

(5) Combining the second and fourth steps to compute the dot products of the partial derivatives with respect to x and y with each other and themselves.

First step. The following formulas for the partial derivatives can be computed directly from the definition of σ given above:

$$\frac{\partial\sigma}{\partial r} = \left(\frac{t}{s}\cos\left(\frac{s\theta}{t}\right), \frac{t}{s}\sin\left(\frac{s\theta}{t}\right), \sqrt{1-\frac{t^2}{s^2}}\right)$$

$$\frac{\partial\sigma}{\partial\theta} = \left(-\frac{tr}{s}\frac{s}{t}\sin\left(\frac{s\theta}{t}\right), \frac{tr}{s}\frac{s}{t}\cos\left(\frac{s\theta}{t}\right), 0\right)$$

Second step. Using the results of the first step we obtain the following formulas for the dot products of the partial derivative functions:

$$\left|\frac{\partial\sigma}{\partial r}\right|^2 = 1\,, \qquad \left|\frac{\partial\sigma}{\partial\theta}\right|^2 = r^2\,, \qquad \frac{\partial\sigma}{\partial r}\cdot\frac{\partial\sigma}{\partial\theta} = 0$$

Third step. We can do this using the Chain Rule and the standard polar coordinate formulas $x = r\cos/theta$, $y = r\sin\theta$:

$$\frac{\partial\sigma}{\partial r} = \cos\theta\frac{\partial\sigma}{\partial x} + \sin\theta\frac{\partial\sigma}{\partial y}\,, \qquad \frac{\partial\sigma}{\partial\theta} = -r\sin\theta\frac{\partial\sigma}{\partial x} + r\cos\theta\frac{\partial\sigma}{\partial y}$$

Fourth step. We can invert the resuls of the prceding step to obtain the following:

$$\frac{\partial\sigma}{\partial x} = \cos\theta\frac{\partial\sigma}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial\sigma}{\partial\theta}\,, \qquad \frac{\partial\sigma}{\partial y} = \sin\theta\frac{\partial\sigma}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial\sigma}{\partial\theta}$$

Fifth step. We can now use the formulas in the second and fourth step to conclude the following:

$$\left|\frac{\partial\sigma}{\partial x}\right|^2 = 1\,, \qquad \left|\frac{\partial\sigma}{\partial y}\right|^2 = 1\,, \qquad \frac{\partial\sigma}{\partial x}\cdot\frac{\partial\sigma}{\partial y} = 0$$

These are the identities we wanted to prove.

Comments

This might seem to be a lot of work to prove something that is pretty obvious, so one natural question is what we can learn from this proof. Here are several points worth noting.

1. One reason for the length and complexity of the argument is the need to have mathematically precise definitions of surfaces and isometries of surfaces.

2. Some calculus texts use the construction in this document to define the surface area of a cone. The derivation points out one drawback of such a definition: It is difficult to justify it using the sorts of ideas that appear in elementary single variable calculus courses or even in subsequent courses in multivariable calculus. The derivation in `notes1100.pdf` was designed to be compatible with the standard approach to surface area problems in multivariable calculus. On the other hand, the result presented above can be viewed as a somewhat informative example for an upper level undergraduate course in the differential geometry of surfaces to illustrate the concept of isometries between different types of surfaces.