Differentiating Negative and Fractional Powers

Known fact: If n is a positive integer, then

$$\frac{d}{dx} x^n = n x^{n-1}.$$

Claim: This generalizes to all real values of n.

Goal: Prove this if n is a rational number.

First Step: Suppose n is a negative integer, so $x^n = \frac{1}{x^m}$ where $m = -n = |n|$.

Apply the rule for differentiating quotients:

$$D\left(\frac{a}{b}\right) = \frac{b \cdot Da - a\,Db}{b^2} \quad \text{where } \begin{cases} a = 1 \\ b = x^m. \end{cases}$$

Substitute $a = 1$, $b = x^m$ into this:

$$D\left(\frac{1}{x^m}\right) = \frac{x^m \cdot \overset{0}{\cancel{D1}} - 1 \cdot m x^{m-1}}{x^{2m}} = \frac{-m}{x^{m+1}} = \boxed{-m x^{-m-1}}.$$

Since $m = -n$, this is just $n x^{n-1}$.

<u>Second step</u>: Every fraction can be written as $\frac{p}{q}$ with $q>0$ (because $\frac{p}{-q} = \frac{-p}{q}$).

Let $y = x^{p/q}$ with $q>0$, and let

$z = y^{q}$, so that $z(x) = x^{p}$.

By the chain rule

$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$. Now $\frac{dz}{dx} = p x^{p-1}$,

and $\frac{dz}{dy} = q y^{q-1} = q\left(x^{p/q}\right)^{q-1} =$

$q\, x^{\left(p - \frac{p}{q}\right)}$, so

$$\frac{dy}{dx} = \frac{\left(\frac{dz}{dx}\right)}{\left(\frac{dz}{dy}\right)} = \frac{p\, x^{p-1}}{q\, x^{(p - p/q)}} =$$

$$\frac{p}{q}\, x^{(p/q - 1)}$$ which is

what we wanted. ∎

<u>Note</u> We need to have information about differentiating exponential and logarithmic functions in order to prove the formula for <u>all</u> n.