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Mathematics 009B–020, Spring 2012, Examination 1

Answer Key

1. [*25 points*] Find the following antiderivative:

$$\int \sqrt[3]{2x-1}\,dx$$

SOLUTION

Let $u = 2x - 1$ so that $du = 2\,dx$ and $du = \frac{1}{2}\,dx$. Then

$$\int \sqrt[3]{2x-1}\,dx \;=\; \int \sqrt[3]{u}\,\tfrac{1}{2}\,du \;=$$

$$\frac{1}{2}\int \sqrt[3]{u}\,du \;=\; \frac{1}{2}\int u^{1/3}\,du \;=\; \frac{1}{2}\,\frac{4}{3}u^{4/3} + C \;=$$

$$\frac{2}{3}(2x-1)^{4/3} + C\,.$$

2. [*25 points*] Find the volume of the solid of revolution generated by rotating the half-ellipse

$$f(x) \;=\; \sqrt{1-\frac{x^2}{4}} \quad , \qquad -2 \;\leq\; x \;\leq\; 2$$

about the x-axis.

SOLUTION

The volume is equal to

$$\pi \int_{-2}^{2} f(x)^2\, dx \;=\; \pi \int_{-2}^{2} \left(1-\frac{x^2}{4}\right) dx \;=$$

$$\pi \left(x - \tfrac{1}{12}\, x^3 |_{-2}^{2}\right) \;=\; \pi \left(2-\frac{8}{12}\right) \;-\; \pi \left(-2-\frac{-8}{12}\right) \;=$$

$$\pi\, \frac{4}{3} \;-\; \pi\, \frac{-4}{3} \;=\; \frac{8\pi}{3}\ .$$

3. [*25 points*] Find the average value of the function $f(x) = |x^5 - 1|$ over the interval $0 \leq x \leq 2$.

SOLUTION

The average value is

$$\frac{1}{2}\int_0^2 |x^5 - 1|\, dx$$

and to evaluate the latter we need to break the integral up into parts over which $x^5 - 1$ is positive and negative. Now $x^5 - 1 \geq 0$ on the interval $[1, 2]$ and $x^5 - 1 \leq 0$ on the interval $[0, 1]$, so the integral we need to compute is equal to

$$\int_0^1 (1 - x^5)\, dx \; + \; \int_1^2 (x^5 - 1)\, dx \quad = \quad x - \tfrac{1}{6}\, x^6|_0^1 \; + \; \tfrac{1}{6}\, x^6 - x|_1^2 \quad =$$

$$\left(1 - \frac{1}{6}\right) \; + \; \left(\frac{64}{6} - 2\right) \; - \; \left(\frac{1}{6} - 1\right) \quad =$$

$$\frac{5}{6} \; + \; \frac{52}{6} \; + \; \frac{5}{6} \quad = \quad \frac{62}{6} \quad = \quad \frac{31}{3}\, .$$

To get the average value, we divide this integral by the length of the interval, which is 2, and find that the average value is

$$\frac{31}{6}\, .$$

4. [*25 points*] Find the work needed to roll up a 10 foot long carpet which weighs 30 pounds and has uniform density. [*Hint:* When x feet of the carpet have been rolled up, the force needed to keep rolling is just the weight of the portion of the carpet that has already been rolled up.]

SOLUTION

When x feet of the carpet have been rolled up, the weight of the rolled up portion of the carpet is (rolled up fraction) $\times$ (total weight), which equals

$$\frac{x}{10} \times 30 \quad = \quad 3x$$

and hence by the observations in the hint the total force needed to keep rolling the carpet is $3x$. Thus the work needed to roll up the carpet is equal to

$$\int_0^{10} 3x\,dx \quad = \quad \frac{3}{2}\,x^2|_0^{10} \quad = \quad 150 \text{ foot} - \text{pounds} .$$