Mathematics 009B–020, Spring 2012, Examination 1

Answer Key

1. [*15 points*] (a) Rewrite $\log_e(a+b) + 5\log_e(a-b) - 3\log_e c$ as $\log_e A$ for a single expression involving a, b, c.

(b) Find

$$\frac{d}{dx}\, 2 \ \arcsin(x^2) \ .$$

SOLUTION

(a) We shall suppress the base subscript from the logarithmic function to simplify the notation. Since $k \log x = \log x^k$ and $\log(1/x) = -\log x$, the expression simplifies to

$$\log(a+b) + 5\log(a-b)^5 - \log c^3 \quad \log\left(\frac{(a+b)(a-b)^5}{c^3}\right) \ .$$

(b) Set $u = x^2$, so that the Chain Rule implies

$$\frac{d}{dx}\, 2 \ \arcsin(x^2) \quad = \quad 2\frac{d}{du} \ \arcsin u \cdot \frac{du}{dx} \quad =$$

$$\frac{2}{\sqrt{1-u^2}} \cdot \frac{du}{dx} \quad = \quad \frac{4x}{\sqrt{1-x^4}} \ .$$

2. [*15 points*] Express $\tan(\arcsin x)$ as an algebraic function of x.

SOLUTION

If $\theta = \arcsin x$, then we want to know what $\tan\theta$ is if we know that $\sin\theta = x$. But we know that

$$\tan\theta \quad = \quad \frac{\sin\theta}{\cos\theta} \quad = \quad \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} \quad = \quad \frac{x}{\sqrt{1-x^2}}$$

and since $\theta = \arcsin x$ it follows that

$$\tan(\arcsin x) \quad = \quad \frac{x}{\sqrt{1-x^2}} \ .$$

3. [*15 points*] Evaluate the indefinite integral

$$\int \frac{-5 \arctan x}{1 + x^2}\, dx \ .$$

SOLUTION

If we make the change of variables $u = \arctan x$, then the integrand can be rewritten as $-5u\, du$. Therefore the indefinite integral is equal to

$$\int -5u\, du \quad = \quad -\tfrac{5}{2}\, u^2 \ + \ C \quad = \quad -\tfrac{5}{2}\, (\arctan x)^2 \ + \ C \ .$$

4. [*15 points*] Evaluate the definite integral

$$\int_0^1 x\, e^{-x}\, dx\ .$$

SOLUTION

Use Integration by Parts as in the case of $x\, e^x$. Specifically, if $u = x$ and $dv = e^{-x}\, dx$, then $v = -e^{-x}$ and Integration by Parts yields the following:

$$\int x\, e^{-x}\, dx \quad = \quad -xe^{-x} \ + \int e^{-x}\, dx \quad = \quad -xe^{-x} \ - \ e^{-x} \ + \ C$$

Therefore the definite integral from $x = 0$ to $x = 1$ is equal to

$$-xe^{-x} \ - \ e^{-x}\,|_0^1 \quad = \quad -2e^{-1} \ + \ 1\ .$$

5. [*20 points*] Evaluate the indefinite integral

$$\int \log_e(2x)\,dx\ .$$

SOLUTION

We shall suppress the base subscript from the logarithmic function to simplify the notation. Since $\log 2x = \log 2 + \log x$, the indefinite integral is just

$$\int \log 2\,dx \ + \int \log x\,dx$$

and using integration by parts on the second summand one checks that this is equal to

$$(\log 2)\,x \ + \ x\log x \ - \ x \ + C$$

since the last three terms are the indefinite integral of $\log x$.

6. [*20 points*] Evaluate the indefinite integral

$$\int \frac{dx}{x^2 - 3x + 2} \ .$$

SOLUTION

The first step is to find the partial fractions expansion of the denominator, which factors as $(x-2)(x-1)$:

$$\frac{1}{x^2 - 3x + 2} \quad = \quad \frac{A}{x-2} \ + \ \frac{B}{x-1}$$

The undetermined coefficients must satisfy $A + B = 0$ and $-A - 2B = 1$, which imply that $B = -1$ and $A = 1$. Therefore we have

$$\int \frac{dx}{x^2 - 3x + 2} \quad = \quad \int \frac{dx}{x-2} \ - \ \int \frac{dx}{x-1} \quad =$$

$$\log(x-2) \ - \ \log(x-1) \ + \ C$$

which can be rewritten as

$$\log\left(\frac{x-2}{x-1}\right) \ + \ C \ .$$