Mathematics 009B–020, Spring 2012, Final Examination

Answer Key

Selected formulas and identities

(Not meant to include everything that might be needed)

Volume for the solid of revolution formed by rotating the region $a \le x \le b$, $0 \le y \le f(x)$ about the x-axis (**Disk Method**):

$$V \;=\; \pi \int_a^b f(x)^2\,dx$$

Volume for the solid of revolution formed by rotating the region $a \le x \le b$, $0 \le y \le f(x)$ about the y-axis (**Shell Method**):

$$V \;=\; 2\pi \int_a^b x\,f(x)\,dx \qquad \text{(assuming } a \ge 0\text{)}$$

Base change rules for logarithms:

$$\log_a x \;=\; (\log_a b)\cdot(\log_b x)\;, \qquad \log_b a \;=\; \frac{1}{\log_a b}$$

Partial fraction expansion in a special case:

$$\frac{x}{(ax+b)^{k+2}} \;=\; \frac{P}{(ax+b)^{k+1}} \;+\; \frac{Q}{(ax+b)^{k+2}}$$

where P and Q are constants to be determined.

First coordinate for the centroid of the region defined by $a \le x \le b$, $g(x) \le y \le f(x)$:

$$x^* \;=\; \overline{x} \;=\; \frac{1}{\text{Area}} \int_a^b x\big(\,f(x) - g(x)\,\big)\,dx$$

Arc length formula for the curve $y = f(x)$, where $a \le x \le b$:

$$L \;=\; \int_a^b \sqrt{1 + f'(x)^2}\,dx$$

Surface area for surface of revolution formed by rotating the curve $y = f(x)$ (where $a \le x \le b$) about the x-axis:

$$S \;=\; 2\pi \cdot \int_a^b f(x)\,\sqrt{1 + f'(x)^2}\,dx$$

Surface area for surface of revolution formed by rotating the curve $y = f(x)$ (where $a \le x \le b$) about the y-axis:

$$S \;=\; 2\pi \cdot \int_a^b x\,\sqrt{1 + f'(x)^2}\,dx \qquad \text{(assuming } a \ge 0\text{)}$$

1. [*25 points*] (a) Find the derivative of $(x^2+1)^{\sin x}$.

(b) Find the derivative of $\log_x e$.

SOLUTION

(a) Use the formula

$$\frac{dy}{dx} \quad = \quad y \cdot \frac{d}{dx} \log y \ .$$

If y is as in the problem, then the right hand side is

$$(x^2+1)^{\sin x} \cdot \frac{d}{dx} \sin x \, \log(x^2+1)$$

and the latter equals

$$(x^2+1)^{\sin x} \cdot \left(\frac{2x \sin x}{x^2+1} \ + \ \cos x \log(x^2+1) \right) \ .$$

(b) By the identities for logarithms on the formula sheet we have

$$\log_x e \quad = \quad \frac{1}{\log x}$$

and the deriviative of the right hand side equals

$$\frac{-1}{(\log x)^2} \cdot \frac{1}{x} \quad = \quad \frac{-(\log_x e)^2}{x} \ .$$

It is not necessary to express the answer in the second form.

2. [*25 points*] Find the area of the region bounded by the parabola $y = x^2$ (lower curve) and the line $y = x + 2$ (upper curve).

SOLUTION

One important step is to find the limits of integration by finding the points at which the two curves meet; in other words, we need to solve the equation $x^2 = x+2$ or equivalently $x^2 - x - 2 = 0$. The roots for this equation are $x = -1$ and $x = 2$.

Therefore the area is given by

$$\int_{-1}^{2} \big((x+2) - x^2 \big)\, dx \quad = \quad \int_{-1}^{2} (2 + x - x^2)\, dx \quad = \quad \frac{-x^3}{3} \;+\; \frac{x^2}{2} \;+ -\; 2x \Big|_{-1}^{2} \quad =$$

$$\left(\frac{-8}{3} - \frac{1}{3} \right) + \left(\frac{4}{2} - \frac{1}{2} \right) + \big(4 - (-2) \big) \quad =$$

$$-3 \;+\; \frac{3}{2} \;+\; 6 \quad = \quad \frac{9}{2}\ .$$

3. [*25 points*] A bucket of cement weighing 50 pounds is being lifted into the air with a 40 foot rope having a uniform density of 0.5 pounds per feet. Compute the work (or energy usage) required to lift the bucket from the ground to a height of 40 feet above the ground.

SOLUTION

The key is to find the force needed to lift everything when the bucket is h feet above the ground. This is 50 pounds plus the weight of the portion of the rope which is hanging down, and since **mass = density × length** the rope weight for height h is equal to

$$0.5 \times (40 - h) \ .$$

Therefore the work done in lifting the bucket is

$$\int_0^{40} 50 + \left(\frac{40-h}{2}\right) dh \quad = \quad \int_0^{40} 70 - \left(\frac{h}{2}\right) dh \quad =$$

$$70h \ - \ \frac{h^2}{4}\Big|_0^{40} \quad = \quad 2800 \ - \ 400 \quad = \quad 2400 \text{ foot} - \text{pounds} \ .$$

4. [*30 points*] Evaluate the following antiderivatives (= indefinite integrals).

(a)

$$\int \sqrt{e^{3x} + e^{2x}}\, dx$$

[*Hint:* Factor the expression inside the radical sign.]

(b)

$$\int \frac{x\, dx}{(x+4)^3}$$

SOLUTION

(a) Following the hint, note that the integrand equals $e^x\sqrt{e^x+1}$, so the integral is

$$\int \sqrt{e^x+1}\, e^x\, dx$$

and if we let $u = e^x + 1$ this becomes

$$\int \sqrt{u}\, du \quad = \quad \frac{2}{3}\, u^{3/2} \;+\; C \quad = \quad \frac{2}{3}\, (e^x+1)^{3/2} \;+\; C\ .$$

(b) We have the partial fraction expansion

$$\frac{x}{(x+4)^3} \quad = \quad \frac{1}{(x+4)^2} \;-\; \frac{4}{(x+4)^3}$$

so that

$$\int \frac{x\, dx}{(x+4)^3} \quad = \quad \int \frac{dx}{(x+4)^2} \;-\; \int \frac{4\, dx}{(x+4)^3}$$

and the latter is just

$$\frac{-1}{(x+4)} \;+\; \frac{-1}{2(x+4)^2} \;+\; C\ .$$

5. [*30 points*] (a) Evaluate the definite integral

$$\int_0^{\pi/2} \cos x \, \sin 2x \, dx \; .$$

(b) Evaluate the improper integral

$$\int_4^{\infty} \frac{x \, dx}{(x^2+1)^2} \; .$$

SOLUTION

(a) Since

$$\sin\alpha\cos\beta \quad = \quad \frac{1}{2}\left(\sin(\alpha+\beta) \; + \; \sin(\alpha-\beta)\,\right)$$

it follows that the integrand equals $\frac{1}{2}(\sin 3x + \sin x)$, so that the integral is

$$\frac{1}{2}\int_0^{\pi/2} (\sin 3x + \sin x)\, dx \quad = \quad \frac{-\cos 3x}{6} \; + \; \frac{-\cos x}{2}\Bigg|_0^{\pi/2} \quad =$$

$$0 \; - \; \left(\frac{-1}{6} + \frac{-1}{2}\right) \quad = \quad \frac{2}{3} \; .$$

(b) The improper integral equals the limit of the ordinary definite integral from 4 to b as $b \to \infty$. To compute

$$\int_4^{b} \frac{x \, dx}{(x^2+1)^2}$$

make the change of variables $u = x^2+1$, so that $x\,dx = \frac{1}{2}\,du$ and this integral equals

$$\frac{1}{2} \cdot \int_{17}^{b^2+1} \frac{du}{u^2} \quad = \quad \frac{1}{2} \cdot \frac{-1}{u}\Bigg|_{17}^{b^2+1} \quad =$$

$$\frac{1}{2} \cdot \left(\frac{1}{17} \; - \; \frac{1}{b^2+1}\right) \; .$$

If we take the limit of this as $b \to \infty$, then the second term disappears and we are left with a final answer of $1/34$.

6. [*20 points*] If D is the region bounded by the graphs of $y = x^2$ and $y = \sqrt{x}$, then D is symmetric with respect to the diagonal line $y = x$ and hence the coordinates (x^*, y^*) of its centroid are equal. Find x^*; you may use the fact that the area of the region D is equal to $1/3$ without proving it.

SOLUTION

The two curves meet when $x = 0, 1$. Therefore by the formula we know that

$$x^* \quad = \quad \frac{1}{\text{area}} \cdot \int_0^1 \ x \cdot (\sqrt{x} - x^2)\, dx$$

and since the area is $1/3$ this is equal to

$$\frac{1}{1/3} \cdot \int_0^1 (x^{3/2} - x^3)\, dx \quad = \quad 3 \cdot \left(\frac{2}{5} x^{5/2} - \frac{1}{4} x^4 \Big|_0^1 \right) \quad =$$

$$3 \cdot \left(\frac{2}{5} - \frac{1}{4} \right) \quad = \quad 3 \cdot \frac{3}{20} \quad = \quad \frac{9}{20} \ .$$

7. [*20 points*] Find (but do not evaluate) an explicit definite integral which gives the arc length for the piece of the hyperbola $y = \sqrt{x^2 - 1}$ where $2 \leq x \leq 3$.

SOLUTION

The formula says that the arc length is the integral of $\sqrt{1 + (y')^2}$ where $y = \sqrt{x^2 - 1}$. The latter implies that

$$y' \quad = \quad \frac{x}{\sqrt{x^2 - 1}} \qquad \text{so that} \quad (y')^2 \quad = \quad \frac{x^2}{x^2 - 1}$$

and hence

$$1 \; + \; (y')^2 \quad = \quad \frac{2x^2 - 1}{x^2 - 1} \; .$$

Therefore the arc length formula implies that the arc length equals

$$\int_2^3 \sqrt{\frac{2x^2 - 1}{x^2 - 1}} \, dx \; .$$

8. [*25 points*] Find the surface area of the spherical surface of revolution obtained by rotating the curve $y = \sqrt{1-x^2}$ about the y-axis, where $0 \leq x \leq \frac{1}{2}$.

SOLUTION

The surface area formula for this example is given by

$$2\pi \cdot \int_a^b x\,\sqrt{1+(y')^2}\,dx$$

where y is as above. In this case we have

$$y' \quad = \quad \frac{-x}{\sqrt{1-x^2}} \ , \qquad \text{so that} \quad (y')^2 \quad = \quad \frac{x^2}{1-x^2}$$

and hence

$$1 \ + \ (y')^2 \quad = \quad \frac{1}{1-x^2} \ .$$

Therefore the surface area formula implies that the surface area equals

$$2\pi \cdot \int_0^{1/2} \frac{x\,dx}{\sqrt{1-x^2}}$$

and if we let $u = 1 - x^2$ (so that $x\,dx = -\frac{1}{2}\,du$) the area formula becomes

$$2\pi \int_1^{3/4} \frac{-du}{2\sqrt{u}} \quad = \quad 2\pi \int_{3/4}^1 \frac{du}{2\sqrt{u}} \quad = \quad 2\pi\sqrt{u}\,\Big|_{3/4}^{1} \quad = \quad 2\pi\left(1 - \frac{\sqrt{3}}{2}\right) \quad =$$

$$\pi(2-\sqrt{3}) \ .$$