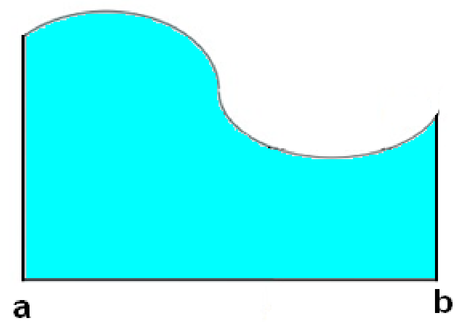
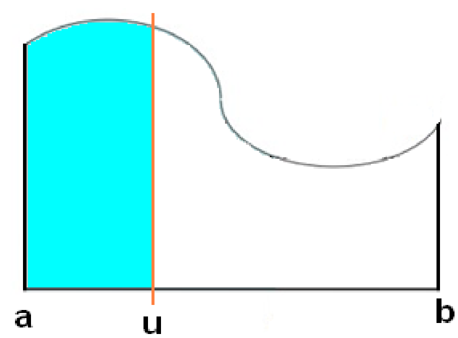
Area under a curve

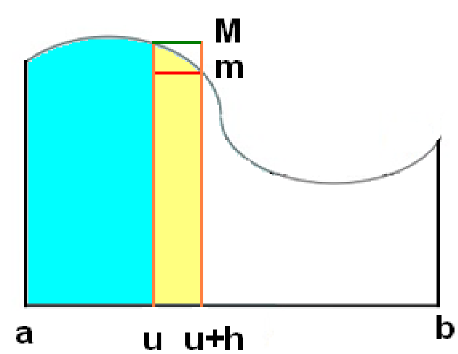
One of the standard problems handled by calculus is calculating the area under the graph of a function $y = f(x)$ which takes nonnegative values. More precisely, if we are given such a function which is defined for all values of x between a and b, we want to find the area of the region in the coordinate plane bounded by the vertical lines $x = a$ and $x = b$, the horizontal line $y = 0$ (the x – axis), and the graph of f, which is defined by $y = f(x)$.



One way trying to compute the area is to look more generally at the area $A(u)$ bounded by the graph, the same horizontal line $y = 0$, the old vertical line $x = a$, and a new vertical line $x = u$ where u can take any value from a to b. Clearly $A(a) = 0$ and $A(b)$ is the area we want.



Now suppose h is a small positive number and consider the difference $A(u+h) – A(u)$. This difference is equal to the area of the thin vertical strip given by all points in the region whose first coordinates lie between u and $u + h$. In the picture on the next page, this is the region colored in yellow.



The area of this region is bounded from below by the area of the rectangular region with width h and length m, where m is the minimum value of the function f between u and $u + h$. Likewise, the area of this region is bounded from above by the area of the rectangular region with width h and length M, where M is the maximum value of the function f between u and $u + h$. In other words, we have

$$\boldsymbol{m \cdot h \;\le\; A(u+h) - A(u) \;\le\; M \cdot h.}$$

If h is positive then we can divide everything in sight by h to obtain a new chain of inequalities

$$\boldsymbol{m \le \frac{A(u+h) - A(u)}{h} \le M}\,.$$

Now if the original function f is continuous then both M and m go to f(u) in the limit as h goes to zero, and therefore we seem to have shown that $A'(u) = f(u)$. We have assumed that h is positive, which means that we have only shown that the limit from the right hand side is f(u), but there is a similar argument which yields the same result if h is negative.

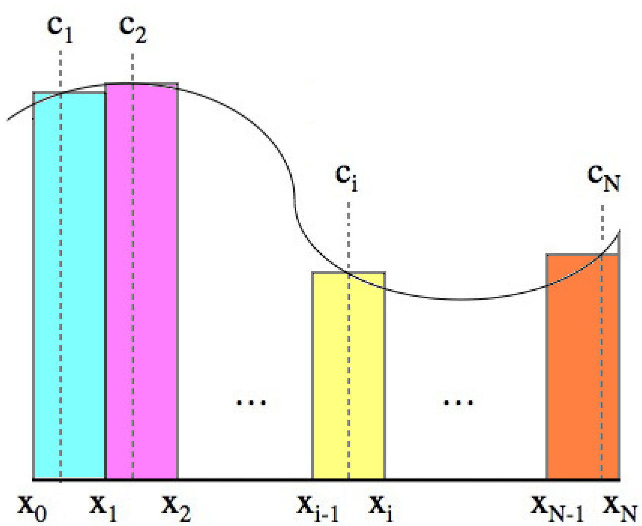
This is essentially one version of the **Fundamental Theorem of Calculus:** ***The area of the region is F(b) – F(a) where F is some function whose derivative is equal to f.***

Another approach

A more systematic way to analyze the area question is to use **Riemann sums.** The basic idea is to cut the entire region into a sequence of vertical strips at points

$$a = x_0 < x_1 < \ldots < x_N = b$$

for some (probably large) value of N, and to estimate the areas of these strips by adding together the areas of suitable rectangular regions whose widths are the successive differences $x_i - x_{i-1}$ (see the next page).



(Source: http://www.askamathematician.com/2011/04/q-why-is-the-integralantiderivative-the-area-under-a-function/)

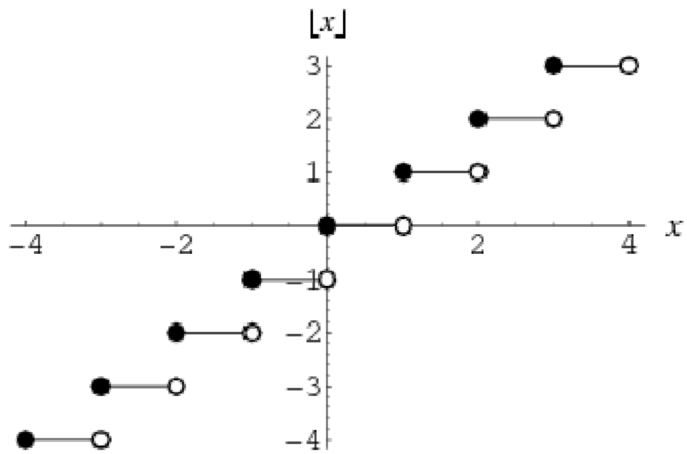
The picture is meant to suggest that the lengths of the various rectangles are given by numbers c_i such that c_i is the value of the function f for some point x between x_{i-1} and x_i. Choosing different values will change the areas of the thin rectangular regions, but in the end this will not matter, for it turns out that the limit of the sums of the areas of these regions (as the maximum width of the rectangular regions goes to zero) will turn out to be the area of the region we are considering. Proofs of these assertions are given in nearly every upper level undergraduate textbook on functions of a real variable.

On a more intuitive note, here are links to two animations which illustrate the fact that the limit of the Riemann sums (as one takes thinner and thinner regions) will be the area under the curve:

http://archives.math.utk.edu/visual.calculus/4/areas.2/area.html

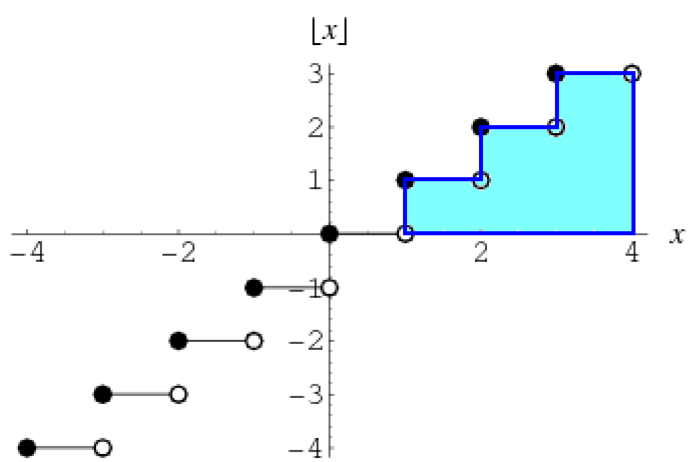
http://archives.math.utk.edu/visual.calculus/4/areas.3/area.html

One advantage of the Riemann sum approach is that it also applies directly to functions that might take negative values or might not be continuous at some points. For example, we might want to consider the area under the graph of the ***greatest integer function*** $f(x) = [x]$.



(Source: http://mathworld.wolfram.com/images/eps-gif/FloorFunction_1000.gif)

Even though this function is not continuous everywhere (the values jump at each integer), it is reasonable to expect that the area under such a curve in the first quadrant, say between $x = 1$ and $x = 4$, corresponds to the area of the bounded staircase region sketched and shaded below, and since this figure is the union of six nonoverlapping squares, its area should be equal to 6. If everything is set up and defined carefully, this turns out to be the case.



Similar sorts of approximating sums also arise in a wide range of other contexts including numerous problems in the sciences and engineering, and ***this is one reason why Riemann sums and integrals receive so much attention in calculus courses.*** Several of these problems will be discussed in the course.

ADDENDUM ON THE USES OF RIEMANN SUMS. The Riemann sum approach to area is also the right one to take if we are given a function f(x) defined empirically by a table of values, with no clear indication of whether or not there is some simply stated formula for the function. Since Riemann sums can only be viewed as approximations to the so – called "true area" of the region under the graph of the function, this approach does not yield a theoretically precise value for the area under the graph, but when a function is defined empirically (say by experiments or observations), then by the nature of the data we cannot expect anything more.

Here is an extremely simple **example**: Suppose we have an object moving along the x – axis in the coordinate plane and we know what its velocities are at specific times. Since the velocity v(t) is the derivative of the position function x(t), it follows that the total distance traveled from time a to time b, which is equal to x(b) – x(a), is given by the integral

$$\boldsymbol{x(b) - x(a) = \int_a^b v(t)dt}$$

but since we only have partial information on v(t) we can only expect to get an approximation to the total distance traveled. We can do this using Riemann sums. Let's assume that the beginning time is 0 on our scale and that the end time is 10 hours later, and that the following table gives the velocity at $\boldsymbol{k}$ hours after the starting time (for convenience, in miles per hour).

$\boldsymbol{k}$	0	1	2	3	4	5	6	7	8	9	10
$\boldsymbol{v(k)}$	53	58	73	80	71	69	55	61	60	59	58

Suppose now that we form Riemann sums as on page 2, such that the function value c_i is equal to the velocity at precisely (i + 1) hours after the start. Then the difference between successive measurements is one hour, and the Riemann sum approximation to the total distance traveled is equal to

58*1 + 73*1 + 80*1 + 71*1 + 69*1 + 55*1 + 61*1 + 60*1 + 59*1 + 58*1 = 644 miles.

Obviously, if we are able to measure the velocity more often, say every half or quarter hour, the approximation to the distance traveled will be more accurate, and we also expect that if we decrease the time intervals between successive measurements the approximations will converge to some limiting value.