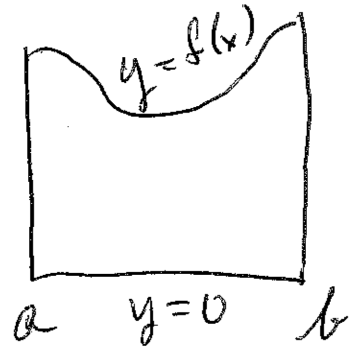
Two definitions of integral

Definite Integral

$$\int_a^b f(x)\,dx$$

If $f \geq 0$,
gives area
under $y = f(x)$



It is given by either

(a) limit of approximating sums,

(b) $F(b) - F(a)$ where

$$F'(x) = f(x).$$

(FUND. THM. CALCULUS)

Indefinite Integral

$\int f(x)\,dx$ no limits a & b

$= F(x) + C$ where $F'(x) = f(x)$.

RULES FOR WORKING WITH INTEGRALS

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1.$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sin x\,dx = -\cos x + C$$

SPECIFICS

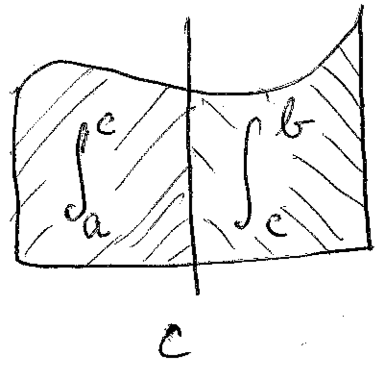
GENERALITIES

$$\text{Linearity}\begin{cases} \displaystyle\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx \\ \qquad\text{(combine constants of integration } C_f,\ C_g \text{ to } C_{f+g}\text{).} \\ \displaystyle\int k f(x)\,dx = k\int f(x)\,dx \\ \qquad\text{(write } kC_f \text{ as } C_{kf}\text{).} \end{cases}$$

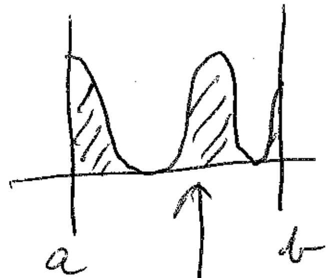
(we only need a simple "$+C$" term).

SPECIAL FOR DEFINITE INTEGALS

$$a < c < b \Rightarrow \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$



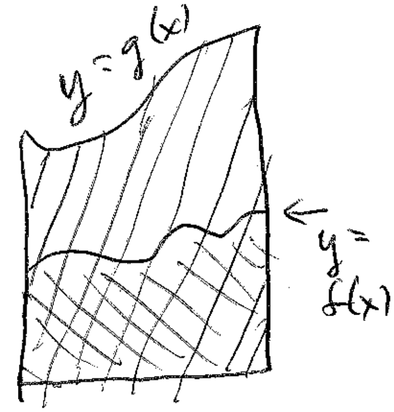
$$0 \le f(x) \Rightarrow \int_a^b f(x)\,dx \ge 0$$



The area under the curve is nonnegative!

<u>Alternate form</u> $f(x) \le g(x)$ all $x \Rightarrow$

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$



Area under $g \ge$ area under f.
(here $f(x) \ge 0$).

<u>Examples</u>

Find $\int x^4 + 3x^3 + 4x\,dx$.

$$\int (x^4 + 3x^3 + 4x)\,dx \; \underline{\underline{\text{Linearity}}}$$

$$\int x^4 dx + 3\int x^3 dx + 4\int x\,dx =$$

$$\frac{x^5}{5} + \frac{3x^4}{4} + \frac{4x^2}{2} \quad \cdots \quad + C$$

$$= \frac{x^5}{5} + \frac{3x^4}{4} + 2x + C$$

Find $\displaystyle\int_{-1}^{1} (|x| - x^3)\,dx$

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x \leq 0 \end{cases}$$

Split into pieces

$$\int_{-1}^{1} (|x| - x^3)\,dx = \int_{-1}^{0} (-x - x^3)\,dx + \int_{0}^{1} (x - x^3)\,dx$$

$$= -\frac{x^2}{2} - \frac{x^4}{4}\Big|_{-1}^{0} + \frac{x^2}{2} - \frac{x^4}{4}\Big|_{0}^{1} =$$

$$\frac{3}{4} + \frac{1}{4} = 1.$$

Change of variables

Find $\int \cos 2x\,dx$.

Trial $\frac{d}{dx}\sin 2x = 2\cos 2x$

so $\frac{d}{dx}\frac{1}{2}\sin 2x = \cos 2x$.

using the chain rule.

More systematically,

if $\frac{d}{dx}y(u(x)) = \frac{dy}{du}\cdot\frac{du}{dx}$, so

$$\boxed{\int f(u(x))\frac{du}{dx}\,dx = \int f(u)\,du}$$

with $u = u(x)$.

Example Find $\int x\cos x^2\,dx$.

<u>Idea</u> Let $u = x^2$

Then $du = \frac{du}{dx}\,dx = 2x\,dx$.

So $x\,dx = \frac{1}{2}du$, and

$$\int x\cos x^2\,dx = \int \cos u \cdot \left(\frac{1}{2}du\right) =$$

$$\frac{1}{2}\int \cos u\,du = \frac{1}{2}\sin u + C \underset{u=x^2}{=}$$

$$\frac{1}{2}\sin x^2 + C.$$

<u>Another example</u>

$$\int x\sqrt{x^2+1}\,dx$$

Again, but
$u = x^2+1$
$du = 2x\,dx$

So we get $\int (u)^{1/2}\,\frac{1}{2}du =$

$$\frac{1}{2}\int u^{1/2}du = \frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{3}u^{3/2}\right) + C =$$

$$\frac{1}{3}\left(x^2+1\right)^{3/2}+C.$$

Try doing

$$\int x^2\sqrt{1+x^3}\,dx \quad \text{similarly}$$

with $u = x^3+1$.

$du = 3x^2dx \qquad x^2dx = \dfrac{du}{3}$

get $\displaystyle\int u^{1/2}\left(\frac{du}{3}\right) = \frac{1}{3}\int u^{1/2}du =$

$$\frac{1}{3}\cdot\frac{2}{3}u^{3/2}+C = \frac{2}{9}\left(x^3+1\right)^{3/2}+C.$$

WARNING There are no simple ways of writing things like $\left\{\begin{array}{l}\displaystyle\int \sin x^2 dx \\ \displaystyle\int\sqrt{1+x^3}\,dx\end{array}\right.$.

One more, with trig identities:

$$\int \sin^3 x \, dx = \int (1-\cos^2 x)\sin x \, dx.$$

set $u = \cos x$, $du = -\sin x \, dx$

RHS becomes $\int (1-u^2)(-du) =$

$$\int (u^2-1)\,du = \frac{u^3}{3} - u + C =$$

$$\frac{\cos^3 x}{3} - \cos x + C.$$

[Idea: Trig identies can be very useful for evaluating integrals!]