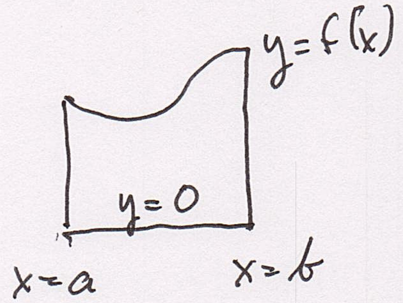
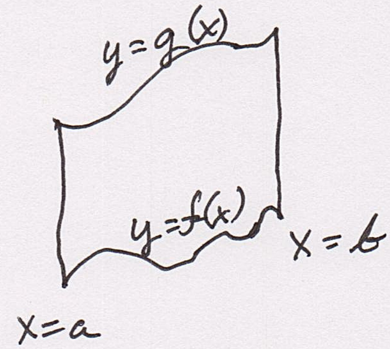
Using (definite) integrals

Area bounded by two curves:

Generalizes area under a curve





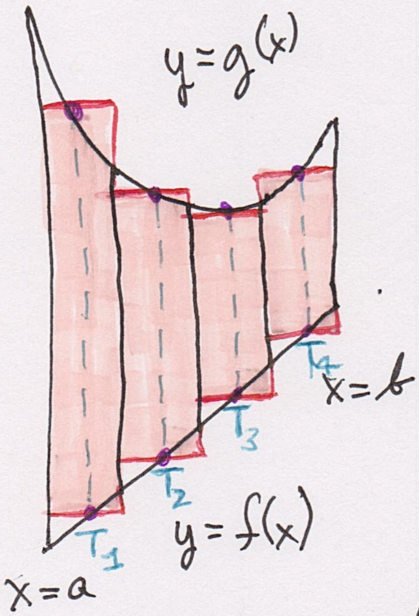
under the graph of $y = f(x) \geq 0$ (and above $y = 0$)

between the graphs $f(x) \leq y \leq g(x)$

ASSUMES $f \leq g$

The ~~definitions~~ derivation of the more general result is similar; one can again approximate the area by Riemann sums. $A = \int_a^b (g(x) - f(x))\,dx$.

EXAMPLE



The area is approximated by

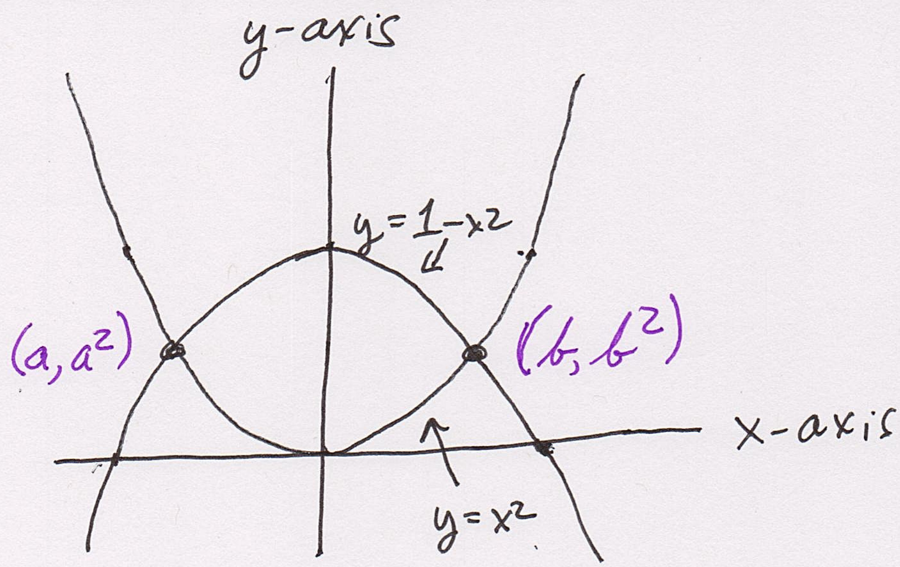
$$\sum \left[g(T_i)-f(T_i)\right]\Delta t,$$

Δt = width of each rectangle, and $T_i \in [x_{i-1}, x_i]$.

$$\Delta t = \frac{b-a}{\#\text{ of intervals}}$$

Approximations get better as Δt gets smaller, and the limiting value as $\Delta t \to 0$ is the area of the region bounded by $\begin{cases} x=a, & x=b \\ y=f(x), & y=g(x). \end{cases}$

Example



$$\text{Area} = \int_a^b \left[(1-x^2) - x^2\right] dx = \int_a^b (1-2x^2)\,dx$$

The key step is to find the points (a, a^2) and (b, b^2) where the curves meet. To do this, solve the eqn. $g(x) = f(x)$:

$$1 - x^2 = x^2$$

$$1 = 2x^2$$

$$\frac{1}{2} = x^2$$

$$\pm\frac{\sqrt{2}}{2} = x.$$

So $\begin{cases} a = -\dfrac{\sqrt{2}}{2} \\ b = +\dfrac{\sqrt{2}}{2} \end{cases}$.

Substitute these into the area formula:

$$A = \int_{-\sqrt{2}/2}^{\sqrt{2}/2} (1 - 2x^2)\, dx =$$

$$x - \frac{2x^3}{3} \Bigg|_{-\sqrt{2}/2}^{\sqrt{2}/2} =$$

$$\left[\frac{\sqrt{2}}{2} - \frac{2}{3}\frac{(\sqrt{2})^3}{2^3}\right] - \left[\frac{-\sqrt{2}}{2} - \frac{2}{3}\frac{(-\sqrt{2})^3}{2^3}\right] =$$

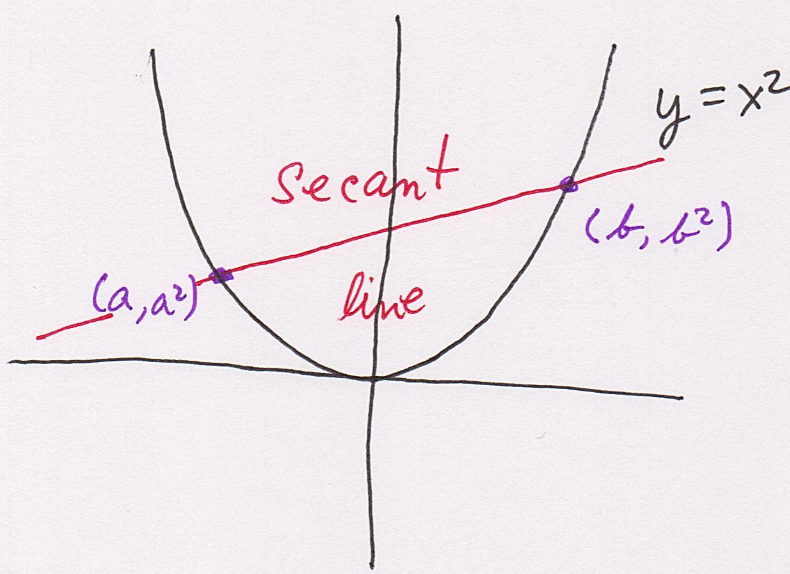
$$\left[\frac{\sqrt{2}}{3} - \frac{2 \cdot 2\sqrt{2}}{3 \cdot 2^3}\right] - \underbrace{\left[\frac{-\sqrt{2}}{3} - \frac{-2 \cdot 2 \cdot \sqrt{2}}{3 \cdot 2^3}\right]}_{\boxed{\text{neg. of 1st term}}} =$$

$$2 \cdot \left[\frac{\sqrt{2}}{3}\left(1 - \frac{1}{2}\right)\right] =$$

$$\frac{\sqrt{2}}{3}.$$

Another example

Find the area of the ~~...~~ region bounded by the parabola $y = x^2$ and the secant line joining (a, a^2) to (b, b^2) with $a \neq b$.



$a < b$.

We need to find the equation of the secant line:

$$\frac{y - a^2}{x - a} = \frac{b^2 - a^2}{b - a} = b + a \quad \text{or}$$

$$y = (b+a)(x-a) + a^2 = (a+b)x - ab.$$

So the area is

$$\int_a^b \left[(a+b)x - ab - x^2\right] dx =$$

$$\frac{(a+b)x^2}{2} - abx - \frac{x^3}{3} \Bigg|_a^b =$$

$$\left(\frac{ab^2+b^3}{2} - ab^2 - \frac{b^3}{3}\right) -$$

$$\left(\frac{a^3+a^2b}{2} - a^2b - \frac{a^3}{3}\right) =$$

$$\cancel{\frac{b^3-a^3}{6} + \frac{a^2b - ab^2}{2}}$$

$$\frac{b^3-a^3}{6} + \frac{a^2b-ab^2}{2}$$

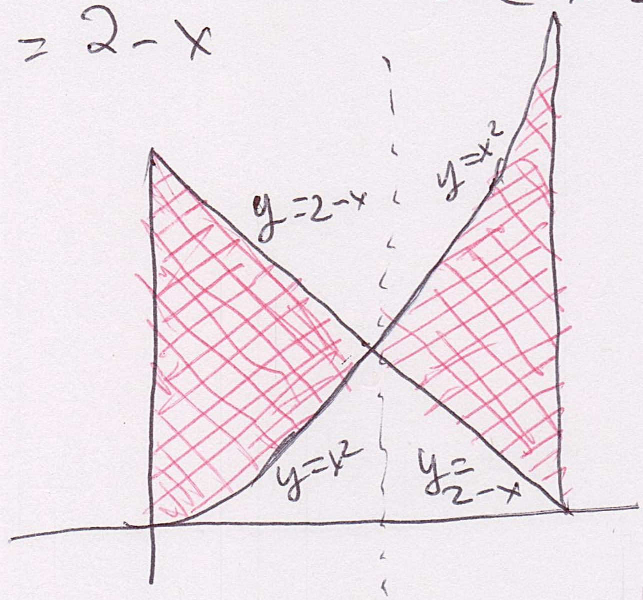
Crossing graphs

Typical example.

$y = x^2$

$y = 2 - x$

$0 \le x \le 2$



First find where graphs cross

Solve $x^2 = 2 - x$

$$x^2 + x - 2 = 0$$

$$(x+2)(x-1) = 0$$

disregard $x = -2$ since it's not between 0 + 2

So $x = 1$

$y = 1$.

So $2-x \geq x^2$ if $0 \leq x \leq 1$

$x^2 \geq 2-x$ if $1 \leq x \leq 2$.

Area = sum of areas =

$$\int_0^1 (2-x-x^2)\,dx + \int_1^2 x^2-(2-x)\,dx =$$

$$\int_0^1 (2-x-x^2)\,dx + \int_1^2 (x^2+x-2)\,dx =$$

$$2x-\frac{x^2}{2}-\frac{x^3}{3}\Big|_0^1 + \left(\frac{x^3}{3}+\frac{x^2}{2}-2x\right)\Big|_1^2 =$$

$$\frac{7}{6} + \left(\frac{8}{3}+\frac{4}{2}-4-\frac{1}{3}-\frac{1}{2}+2\right) =$$

$$\frac{7}{6} + \left(\frac{7}{3}+\frac{3}{2}-2\right) =$$

$$\frac{7}{6}+\frac{14}{6}+\frac{9}{6}-\frac{12}{6}=\frac{18}{6}=\boxed{3.}$$