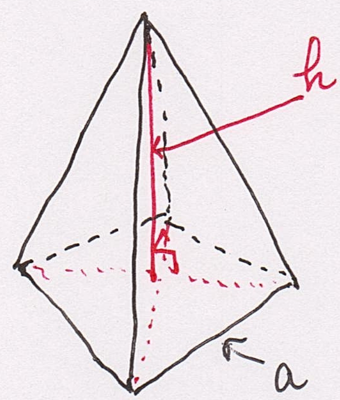
Using integrals to compute volumes

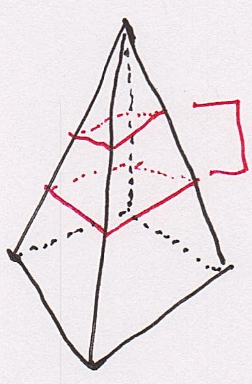
First nontrivial examples

red line is through top vertex, $\perp$ base at its center



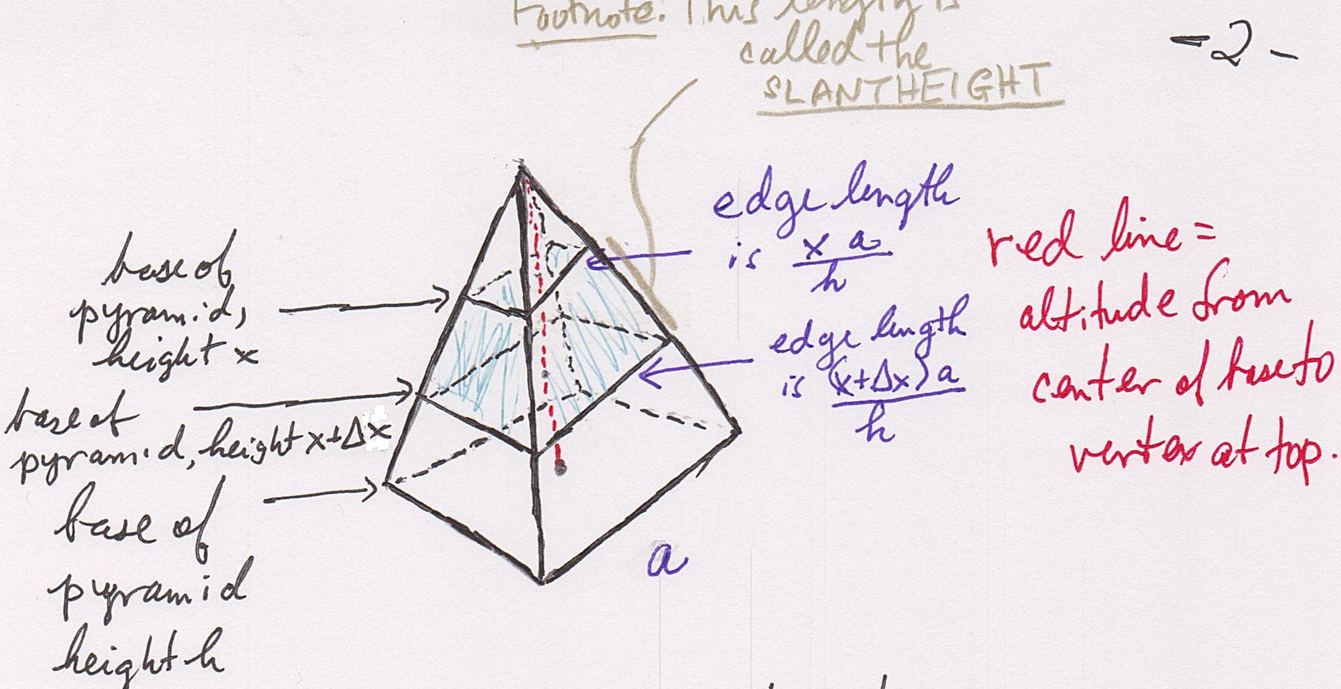
Square pyramid height h base has edge of length a.

Approach Slice pyramid horizontally into n pieces of thickness $\frac{h}{n} = \Delta x$



SLICE, THICKNESS = Δx

Get upper and lower estimates for volume between $z = x_i$ and $z = x_{i+1} = x_i + \Delta x$



Want to estimate the volume of the piece between the $\parallel$ planes

$z = x$ and $z = x + \Delta x$

rect solid with height Δx, square base with edge of length $\frac{x\,a}{h}$ $\leq$ VOL. $\leq$ rect solid with height Δx, square base with edges of length $\frac{(x+\Delta x)\,a}{h}$

$$\Downarrow$$

$$\frac{x^2 a^2}{h^2}\,\Delta x \leq \text{VOL} \leq \frac{(x+\Delta x)^2 a^2}{h^2}\,\Delta x$$

So we can approximate the volume by the following Riemann sum:

$$\sum \frac{x_i^2 a^2}{h^2} \Delta x$$

As $n \to \infty$ and $\Delta x \to 0$, the limit value equals $\frac{a^2}{h^2}\int_0^h x^2 dx =$

$$\frac{a^2}{h^2}\frac{x^3}{3}\Big|_0^h = \boxed{\frac{1}{3} a^2 h}$$

standard formula from geometry.

A similar argument work for cones, with the slice volume between that of two cylinders whose volumes are

$$\frac{\pi x^2 r^2}{h^2}\Delta x \quad \text{and} \quad \frac{\pi (x+\Delta x)^2 r^2}{h^2}.$$

The volume is then

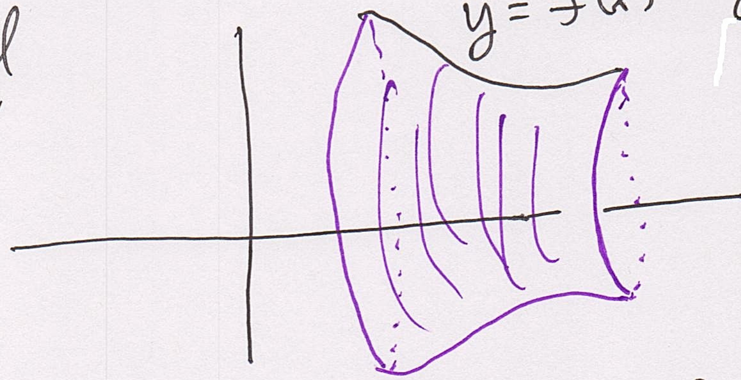
$$\frac{\pi r^2}{h^2}\int_0^h x^2 dx = \boxed{\frac{1}{3}\pi r^2 h.}$$

Solids of revolution

Two types of problems

Disk method

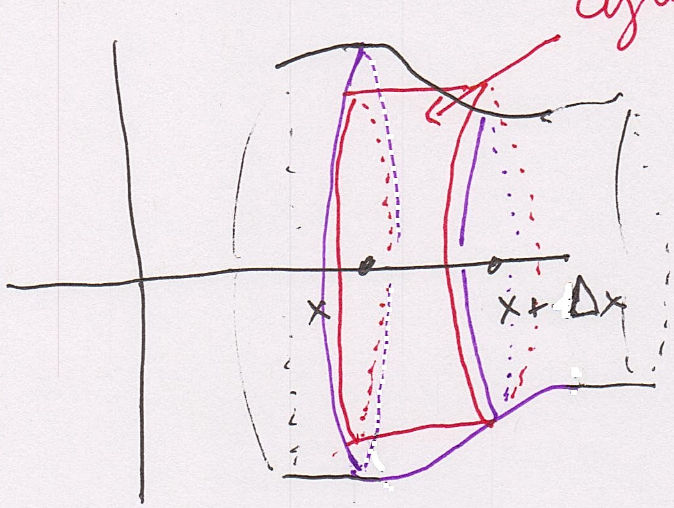
$y = f(x) > 0 \quad a \le x \le b$



$y^2 + z^2 = f(x)^2$ eqn.

Estimate the volume by slicing into thin pieces and estimating the volume of each one with the volume of some disk

cylinder, height Δx & radius $f(C)$, where C is between x and $x + \Delta x$.



In this case the appropriate Riemann sum approximations are

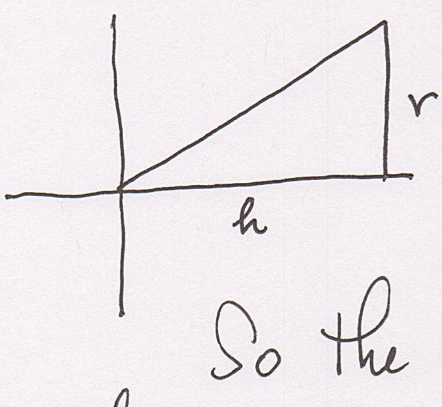
$$\sum \pi f(c_i)^2 \Delta x$$

and the limit of these sums is

$$\pi \int_a^b f(x)^2 dx.$$

Examples

Back to circular cone.



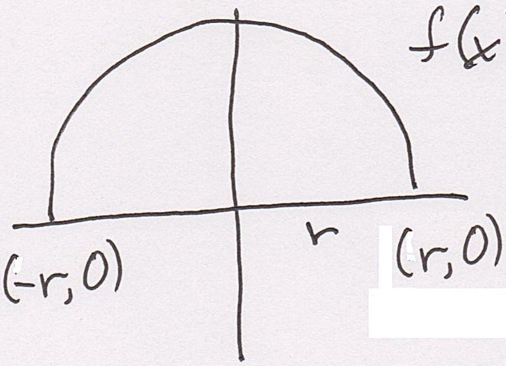
$f(x)$ is the linear function

$$f(x) = \frac{r}{h} x.$$

So the volume is equal to

$$\pi \int_0^h \frac{r^2}{h^2} x^2 dx = \frac{\pi r^2}{h^2} \int_0^h x^2 dx = \frac{\pi r^2}{h^2} \cdot \frac{x^3}{3}\Big|_0^h = \frac{\pi r^2 h}{3}.$$

Sphere



$f(x) = \sqrt{r^2 - x^2}$

upper semicircle

$$V = \pi \int_{-r}^{r} (r^2 - x^2)\, dx$$

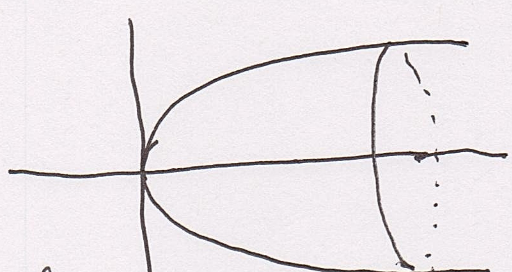
$$= \pi r^2 x \Big|_{-r}^{r} - \frac{\pi x^3}{3} \Big|_{-r}^{r} =$$

$$2\pi r^3 - \frac{2\pi r^3}{3} = \frac{4}{3}\pi r^3.$$

Elliptic paraboloid segment

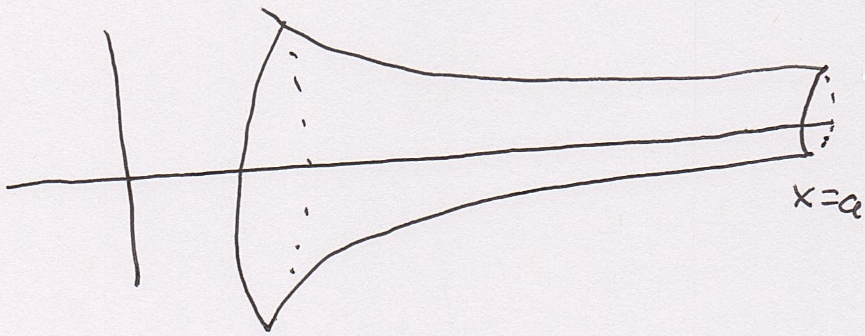
$f(x) = r\sqrt{x}$, $r > 0$

$0 \le x \le h.$



$$V = \pi \int_0^h r^2 x\, dx = \pi r^2 \cdot \frac{x^2}{2} \Big|_0^h = \boxed{\frac{\pi r^2 h^2}{2}}.$$

Torricelli's / Gabriel's stuffed trumpet



$f(x) = \frac{1}{x}$

$1 \leq x \leq a$

$$V = \pi \int_1^a \frac{1}{x^2}\,dx = \pi \cdot \left.\frac{-1}{x}\right|_1^a =$$

$\pi \cdot \left(1 - \frac{1}{a}\right)$. Notice that $V \to \pi$ as $a \to \infty$ (Torricelli)

Later we shall see that the area of the boundary goes to ∞ as $a \to \infty$.