Contrast this with what happens if we take $f(x) = \frac{1}{\sqrt[3]{x}}$. $\quad 1 \le x \le a$.

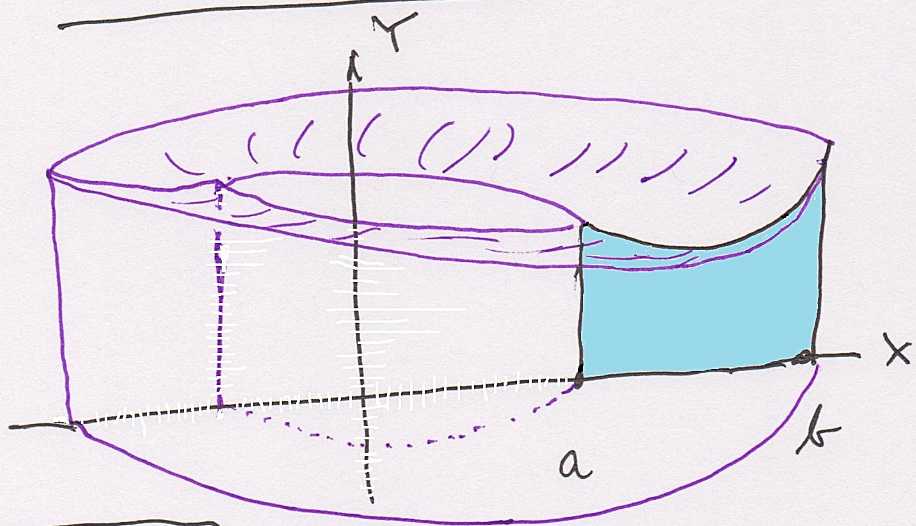
$$V = \pi \int_1^a f(x)^2 \, dx = \pi \int_1^a x^{-2/3} \, dx =$$

$$\pi \cdot \frac{1}{1/3} \cdot x^{1/3} \Big|_1^a =$$

$$3\pi \sqrt[3]{x} \Big|_1^a = 3\pi \left(\sqrt[3]{a} - 1 \right).$$

As $a \to \infty$, $V \to \infty$.

Shell method



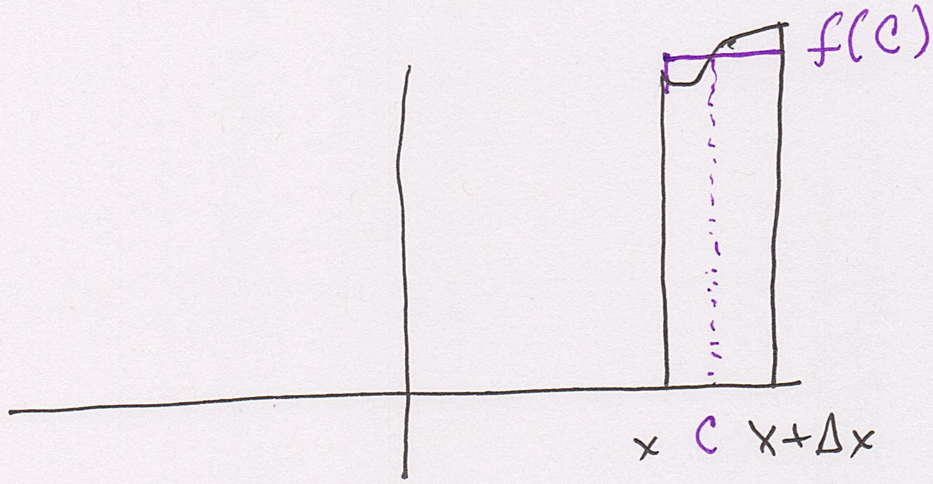
$y = f(x) \geq 0$
$0 < a$.

> Think of the z-axis as $\perp$ to the xy plane.

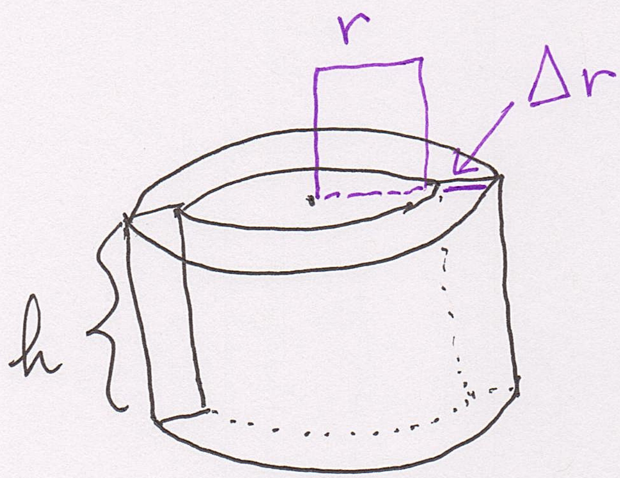
EQN. $y = f(\sqrt{x^2+z^2})$.

Idea: Estimate by splitting into n cylindrical shells of thickness $\Delta x = \frac{b-a}{n}$,

estimate the volumes of these using volumes of cylinders, take limits as $\begin{cases} n \to \infty \\ \Delta x \to 0. \end{cases}$



Find the volume of the cylindrical shell which approximates the shell of thickness Δx.



Cylindrical shell volume

$$= \pi(r+\Delta r)^2 h - \pi r^2 h$$

$$= \pi h\left[(r+\Delta r)^2 - r^2\right] =$$

$$= \pi h\left[(2r+\Delta r)\cdot(\Delta r)\right].$$

$$= 2\pi h r \Delta r + \pi h \Delta r^2.$$

Get approx. to total volume by adding these together!

$$V \approx \sum \underset{=h}{f(c_i)} \cdot (2\pi x_i \Delta x + \pi \Delta x^2)$$

here $x_i = r$

$$= \sum 2\pi f(c_i) x_i \Delta x +$$

$$\sum \pi f(c_i) \Delta x^2 .$$

The first term looks like an approx. to an integral by something like a Riemann sum. In fact, as $\left\{ \begin{matrix} n \to \infty \\ \Delta x \to 0 \end{matrix} \right\}$ the limiting value is just $\int_a^b 2\pi x f(x)\,dx$.

Next, in the limit the approximations to V go to the latter. So we almost have $V = 2\pi \int_a^b x f(x)\,dx$, but to conclude this we need to check that the limit of the second sum

$$\pi \sum f(c_i) \Delta x^2$$

is zero.

<u>RECALL</u> $\Delta x = \frac{b-a}{n}$.

We then have that the second sum is at most $\pi \sum_{i=1}^{n} M \cdot \frac{(b-a)^2}{n^2}$ where

M = max value of f on $[a, b]$, and the latter is in turn at most

$$\pi \cdot (\#\text{terms}) \frac{(b-a)^2}{n^2} \cdot M =$$

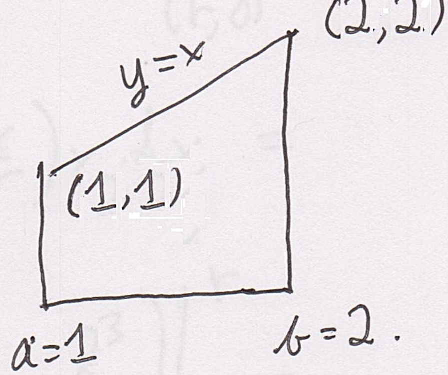
$$\pi \frac{(b-a)^2}{n} M$$ since there are

n terms in the sum. Since $f(c_i) \geq 0$ always, we have

$$0 \le \pi \sum f(c_i)\Delta x^2$$
$$\le \frac{\pi M (b^2-a)^2}{n}$$

which goes to 0 as $n \to \infty$.

Examples.

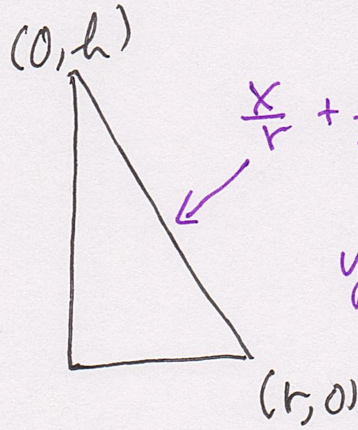


$$V = 2\pi \int_1^2 x\, f(x)\, dx = 2\pi \int_1^2 x^2\, dx =$$

$$\frac{2\pi}{3} x^3 \Big|_1^2 = \frac{14\pi}{3}.$$

Old example, new approach

Cone



$\frac{x}{r} + \frac{y}{h} = 1$

or

$y = h\left(1 - \frac{x}{r}\right).$

$$V = 2\pi h \int_0^r \left(1 - \frac{x}{r}\right) x \, dx. =$$

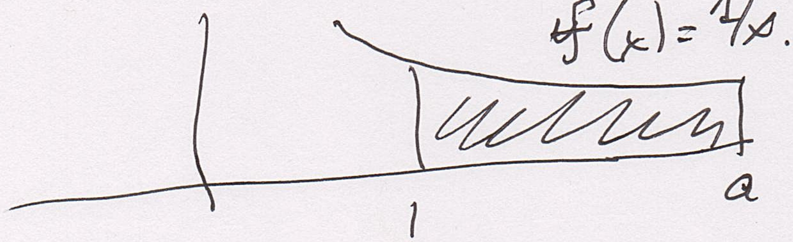
$$2\pi h \left(\frac{x^2}{2} - \frac{x^3}{3r}\right)\Bigg|_0^r =$$

$$2\pi h \left(\frac{r^2}{2} - \frac{r^2}{3}\right) = 2\pi h \frac{r^2}{6} =$$

$$\frac{1}{3}\pi r^2 h.$$

Hyperbola

$$y = \frac{1}{x} \qquad 1 \le x \le a.$$



$$V = 2\pi \int_1^a x f(x)\,dx = 2\pi \int_1^a 1 \cdot dx =$$

$$2\pi(a-1).$$

Graph of x^n $\quad n \ge 3,\ 0 \le x \le 1$

$$V = 2\pi \int_0^1 x \cdot x^n\,dx =$$

$$2\pi \left.\frac{x^{n+2}}{n+2}\right|_0^1 = \frac{2\pi}{n+2}.$$