Work (energy usage) integrals

As before, there are 2 key points:

Specific Integral formulas in physics.

General Recognizing ways in which integrals arise "in nature."

BASIC PHYSICS

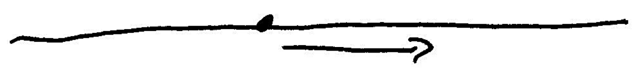
Energy = ability to do work (PHYSICAL DEF.)

Work in a simple case: If the motion is 1-dimensional and the force is constant, then

$$\text{work} = (\text{force}) \times (\text{distance}).$$

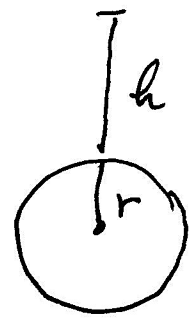
(Generalize to 2D and 3D using vectors.)

$F(x)$



<u>More complicated situation</u>: Motion in a line but force variable.

<u>EXAMPLE</u>

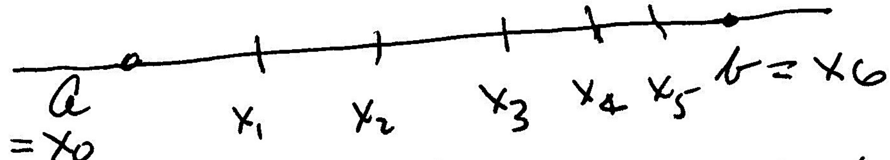


Rocket launching. Find work needed to shoot rocket h units upward. Planet radius $= r$.

If $0 \le x \le h$, then Newton's Law of Gravitation $\Rightarrow$ necessary force

$$F(x) = \frac{K}{(x+r)^2} \qquad (K = \text{some constant})$$

<u>General derivation</u>



Suppose force is constant F_i between x_{i-1} and x_i. Total work =

Sum of work over the individual intervals

$$= \sum F_i \underset{\substack{\uparrow \\ = x_i - x_{i-1}.}}{\Delta x_i}$$

Now suppose force $F(x)$ is continuous. If we take all the intervals to be very small, $F(x)$ will be nearly constant on each interval. Hence work is approx. by the Riemann sum

$$\sum F(c_i)\Delta x_i \qquad c_i \text{ between } x_{i-1} \,\&\, x_i$$

Once again, approximations get better as the maximum subinterval length gets smaller, and the limiting value as the max length $\to 0$ is

(a) the "true value" of the work,

(b) the definite integral $\int_a^b f(x)dx$.

Apply this to the rocket problem

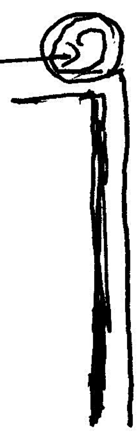
$$\text{Work} = \int_0^h \frac{K\,dx}{(x+r)^2} \underset{u=x+r}{=\!=}$$

$$\int_r^{r+h} \frac{K\,du}{u^2} = \left.\frac{-K}{u}\right|_r^{r+h} =$$

$K\left(\frac{1}{r} - \frac{1}{r+h}\right)$. Note that the limit as $h \to \infty$ is just $\frac{K}{r}$.

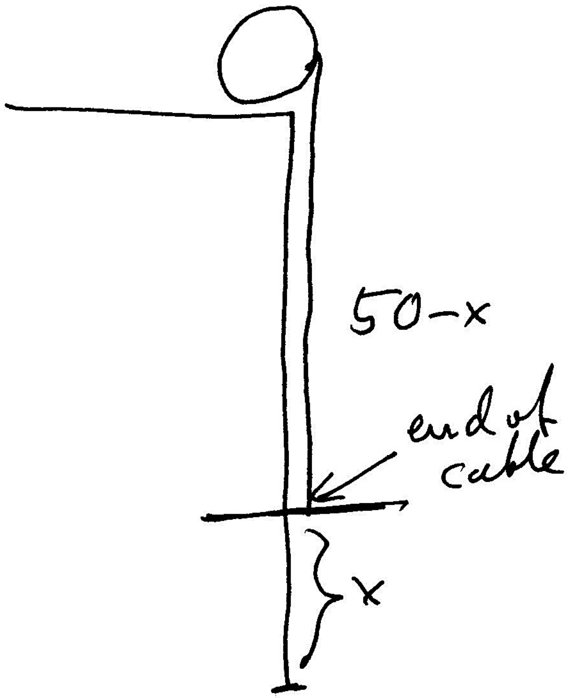
ANOTHER EXAMPLE.

spool



cable: uniform density, 50 ft. hanging, weight = 75 lb.

Find work done in winding the cable onto the spool.



How much force is needed when the end of the cable has been raised x ft?

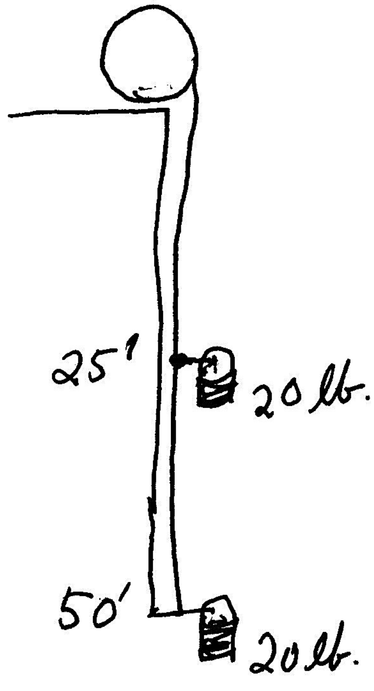
$(50-x)$ feet are still hanging, so the weight is $\left(\frac{50-x}{50}\right)\cdot 75$

$= \frac{3}{2}(50-x)$ lb.

$$\text{Work} = \int_0^{50} F(x)\,dx = \int_0^{50} \frac{3}{2}(50-x)\,dx =$$

$$\frac{3}{2}\cdot\left(50x-\frac{x^2}{2}\right)\Big|_0^{50} = \frac{3}{4}\cdot 2500 = 1875 \text{ ft}\cdot\text{lb.}$$

One can modify this to get a work problem where $F(x)$ is <u>not</u> continuous.



Suppose we have buckets of cement at the end of the cable and halfway up. Set up the work integral for this problem if the buckets each weigh 20 lb.

$$F(x) = \begin{cases} \frac{3}{2}(50-x) + 40 & 0 \le x \le 25 \text{ (two buckets hanging)} \\ \frac{3}{2}(50-x) + 20 & 25 < x \le 50 \text{ (only one bucket)} \end{cases}$$

{jump in values at $x=25$}

THEN

$$W = \int_0^{25} \left[\frac{3}{2}(50-x) + 40\right] dx + \int_{25}^{50} \left[\frac{3}{2}(50-x) + 20\right] dx.$$