Transcendental Functions

[Don't satisfy an identity of the form $\sum c_{ij} x^i f(x)^j = 0$ where not all $c_{ij} = 0$].

So algebraic means $\sum c_{ij} x^i f(x)^j = 0$ where the c_{ij} are not all 0.

algebraic

not transcendental

$$f(x) = \frac{1}{x^2+1}$$

$$f(x)(x^2+1) - 1 = 0.$$

polynomials are algebraic:

$$f(x) - \sum a_i x^i = 0.$$

all rational functions are algebraic.

$$f(x) = \frac{\sum a_i x^i}{\sum b_j x^j} \qquad \begin{array}{l} a_i \text{ not all } 0 \\ b_j \text{ not all } 0. \end{array}$$

radical functions are algebraic

For example if $f(x) = \sqrt[3]{x}$ then

$$f(x)^3 - x = 0. \quad , \quad f(x) = \sqrt{x^2+1} \Rightarrow$$

$$f(x)^2 - x^2 - 1 = 0.$$

Trigonometric and exponential functions are transcendental. (Need more than 1st yr. calculus to prove this. Methods from Math 46, 146 sequence needed. See the course directory file transcendentals.pdf.).

OBJECTIVES OF CHAPTER 9

$\boxed{\text{I}}$ Find antiderivative of $\frac{1}{x}$.

$\boxed{\text{II}}$ Differentiate the $\frac{\text{exponential}}{\text{power}}$ fcn. a^x, where $a > 1$. Likewise for

$\log_a x$

$\arcsin x$, $\arctan x$, $\arccos x, \ldots$

<u>Note</u>: $\arcsin x$ is the unique θ such that $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $x = \sin\theta$. This is only defined if $-1 \le x \le 1$.

Need to analyze inverse functions

$y = f(x)$ inverse function means

$f(x)$ is 1-1: $x \neq x' \Rightarrow f(x) \neq f(x')$.

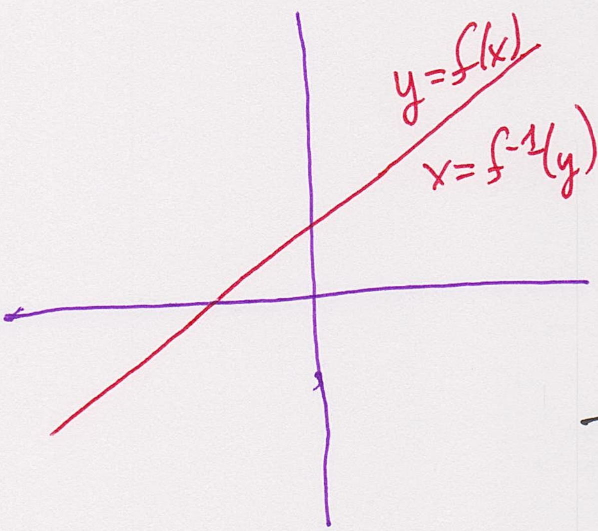
$x = g(y) \Longleftrightarrow y = f(x)$

$g = f^{-1}$

need f to be 1-1 in order for this to work. If $f(x_1) = f(x_2) = y$ but $x_1 \neq x_2$, have two choices for $g(y)$. Functions are single-valued.

Examples

linear functions $y = ax + b = f(x)$
$a \neq 0$



Notice
$x < x' \Rightarrow$
$f(x) < f(x')$
so $f(x)$ is 1-1.

To find $g = f^{-1}$, solve for y in terms of x:

$$x = \frac{y}{a} - \frac{b}{a}$$

Note that $g(f(x)) = x$,

for $x = g(y) \Longleftrightarrow y = f(x) \Longrightarrow$

$x = g(f(x))$. Likewise, $f(g(y)) = y$.

VERIFY THESE WHEN $f(x) = ax + b$,

WHERE $a \neq 0$.

<u>Other examples</u>

$f(x) = x^n$ $\quad f^{-1}(x) = \sqrt[n]{x}$ $\begin{cases} \text{all } x, \ n \text{ odd} \\ x \geq 0, \ n \text{ even} \\ (\text{no real } \sqrt{-1}). \end{cases}$

$f(x) = 10^x$ $\quad f^{-1}(x) = \log_{10} x$.

$f(x) = \sin x$ $\quad f^{-1}(x) = \arcsin x$ $\quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

$f(x) = \cos x$ $\quad f^{-1}(x) = \arccos x$ $\quad 0 \leq x \leq \pi$.

↑

need a range where f is 1–1.

Calculus and inverse functions

(1) Test when a function has an inverse.

(2) Derivative of f^{-1} in terms of the derivative of f (if f' exists).

Strictly increasing / decreasing functions

provided $f'(x)$ always positive / negative

Conversely, f' exists & f is 1-1 $\Rightarrow$ f incr / decr.

Example $f(x) = x^3$ 1-1 increasing

but $f'(0) = 0$.

Inverse Function Theorem

$f' \neq 0$,

$$g = f^{-1} \Rightarrow g'(y) = \frac{1}{f'(g(y))}$$

easier to remember $\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)}$.

Proof Use Chain Rule:

$$y = f(g(y)) \Rightarrow$$

$$1 = \frac{d}{dy} f(g(y)) = f'(g(y)) \cdot g'(y)$$

Now $f' \neq 0$ everywhere, so we can divide by $f'(g(y))$.

<u>WARNING</u> It is not always possible to find a decent formula for $f^{-1}(x)$ even if f is a fairly simple looking polynomial.

<u>Book's example</u> x^5+x^3+x+1

<u>IMPACT</u> Often $f^{-1}(x)$ can only be studied numerically.

<u>Typical problem:</u> Let $f(x)=x^3+x$. Prove f has an inverse g, and find $g'(2)$.

MONIC: TOP DEG TERM HAS COEFF $=1$

<u>Solution</u> $\overset{\text{Monic}}{\overline{\text{odd}}}$ degree polys satisfy $\lim_{x\to\pm\infty} f(x)=\pm\infty$. Check f is increasing

$$f'(x)=3x^2+1\geq 1 \Rightarrow f'(x)>0 \text{ all } x.$$

Hence g exists (but there is no simple formula).

Find $g'(2)$.

$$g'(2) = \frac{1}{f'(g(2))}$$

Need to find $g(2) = x$, or solve $2 = x^3 + x$. Easy in this case: $x = 1$.

Hence $g'(2) = \dfrac{1}{f'(1)}$.

But $f'(1) = 3x^2 + 1\Big|_{x=1} = 4$.

Hence $g'(2) = \dfrac{1}{4}$.

Abstract "example"

$$y = f(x) = x^2 \qquad x > 0$$

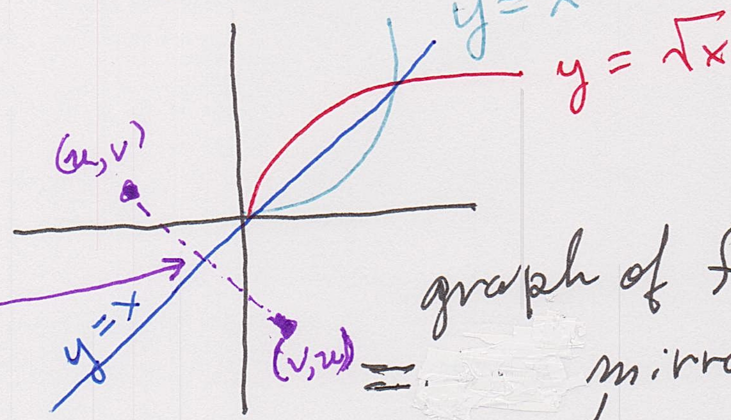
$$g(y) = \sqrt{y} = x. \qquad g'(y) = \frac{d}{dy}\sqrt{y}.$$

$$g'(y) = \frac{1}{f'(g(\cancel{x}y))}$$

$$= \frac{1}{2 \cdot g(y)} = \frac{1}{2\sqrt{y}}, \text{ which}$$

is the same answer we obtained from the formula for $\frac{d}{dy} y^{1/2}$.

$\boxed{\text{VISUAL}}$



(u, v) and (v, u) are mirror images of each other.

graph of $f^{-1}(x)$ = mirror image of graph of $f(x)$ with respect to line $x = y$