The formula $\frac{d}{dx}\log_a x = \frac{K}{x}$

(K = some constant)

Direct assault is very slow going

Football analogy Like a run up the middle of the line. In this case an "end run" is better.

Approach Try to analyze the antiderivative of $\frac{1}{x}$.

$x \geq 1$

$$L(x) = \int_1^x \frac{du}{u} \quad \text{by definition.}$$

$$(\Rightarrow L(1) = 0)$$

Key observation

$$L(xy) = L(x) + L(y)$$

Idea Hold y constant.

$$\frac{d}{dx}L(xy) = \frac{dL}{d(xy)} \cdot \frac{d(xy)}{dx} = \frac{1}{xy} \cdot y = \frac{1}{x}.$$

So $\frac{d}{dx} L(xy) = \frac{d}{dx} L(x) = \frac{1}{x}$

and $L(x \cdot y) = L(x) + C$ (constant)

Set $x = 1$. Then we get $C = L(y)$.

Consequences:

$$L(x^n) = nL(x) \quad n \text{ integer} \geq 0$$

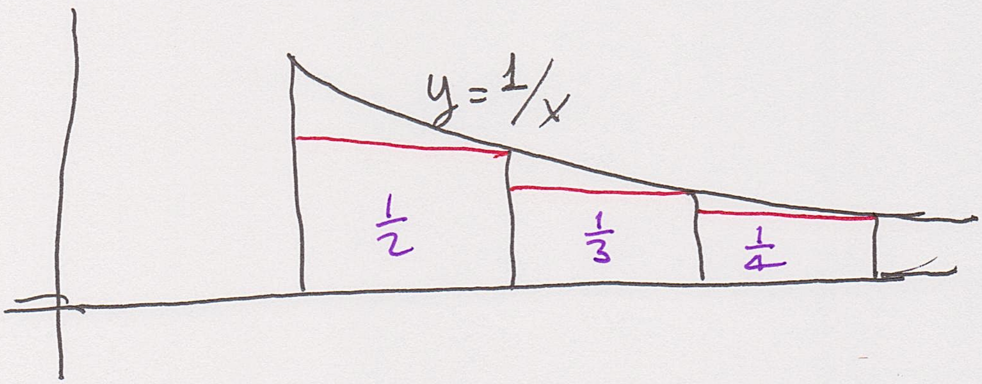
$$L(x^{1/n}) = \frac{1}{n} L(x) \quad n \text{ integer} > 0$$

$$L(x^{p/q}) = \frac{p}{q} L(x) \quad p, q \text{ pos integers.}$$

By continuity

$$L(x^r) = r\,L(x) \quad r \text{ any positive real number.}$$

CLAIM There is a unique e s.t. $1 < e < 4$ and $L(e) = 1$.



Picture indicates that

$$L(4) = \int_1^4 \frac{du}{u} \geq \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12} > 1.$$

Intermediate value property $\Rightarrow$ $L(x) = 1$ someplace between 1 & 4.

$L'(x) = \frac{1}{x} > 0 \Rightarrow$ only one such point.

CONSEQUENCE If $y = L(x)$, then $x = e^y$, so that $y = \log_e x \begin{cases} \log x \\ \ln x \end{cases}$

NATURAL LOGARITHM

This works when $x \geq 1$. For $0 < x \leq 1$ consider $-L(1/x)$. Claim $f(x) = -L(1/x)$

$$\Rightarrow \quad f(x) = \log_e x$$

$$f'(x) = \frac{1}{x}.$$

Details for the sake of completeness

$$f(x) = -\int_1^{1/x} \frac{du}{u} = -\log_e(1/x) = \log_e(x).$$

$$\frac{d}{dx} f(x) = -\frac{d}{dx} L(1/x) = -\frac{dL(1/x)}{d(1/x)} \cdot \frac{d(1/x)}{dx} =$$

$$-\frac{1}{(1/x)} \cdot \left(\frac{-1}{x^2}\right) = \frac{1}{x}.$$

CONSEQUENCE $\quad f(x) > 0$ everywhere $\Rightarrow$

$$\frac{d}{dx} \log(f(x)) = \frac{f'(x)}{f(x)}.$$

What is $\frac{d}{dx}\log_a x$?

Chain Rule for Logarithms

$$\log_a x = \log_a e \cdot \log_e x$$

$$x = a^{\log_a x} = e^{\log_e x} = \left(a^{\log_a e}\right)^{\log_e x} \underset{\text{LAW}}{\overset{\text{EXP.}}{=}}$$

$a^{\log_a e \log_e x}$. Since a^u is a strictly increasing function of u, we have

$$\log_a x = \log_a e \cdot \log_e x$$

$$\Longrightarrow \frac{d}{dx}\log_a x = \frac{\log_a e}{x} = \frac{1}{(\log_e a)\cdot x}$$

Note that $\log_e a = \frac{1}{\log_a e}$ by the Chain Rule for Logarithms.

Examples 1. Compute

$$\int \frac{x\,dx}{x^2+1}$$

Let $u = x^2+1$
$du = 2x\,dx$, so

$\uparrow$ equals $\int \frac{1}{u} \cdot \frac{du}{2} =$

$$\frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\log u + C = \frac{1}{2}\log(x^2+1) + C.$$

2. Differentiate $\log(x^4+x^2)$

$$\frac{d}{dx}\log(x^4+x^2) = \frac{1}{x^4+x^2}\frac{d}{dx}(x^4+x^2) =$$

$$\frac{4x^3+2x}{x^4+x^2}.$$

3. Differentiate $u(x)^{v(x)}$

Use the logarithmic differentiation rule

$$\frac{d}{dx}\log f = \frac{1}{f} f', \text{ so that}$$

$$\frac{df}{dx} = f \cdot \frac{d}{dx}\log f.$$

Here $f = u^v$,

$\log f = v \log u$

So $\frac{d}{dx}(u^v) = u^v \frac{d}{dx}(v \log u) =$

$$u^v\left(\frac{dv}{dx}\log u + v\frac{d}{dx}\log u\right) =$$

$$u^v\left(\frac{dv}{dx}\log u + v \cdot \frac{1}{u}\frac{du}{dx}\right) =$$

$$\log u \cdot u^v \frac{dv}{dx} + v u^{v-1}\frac{du}{dx}.$$

Special case: $u = v = x$

$$\frac{d}{dx} x^x = \log x \cdot x^x + x^x.$$

An easier one:

$$\frac{d}{dx} a^x = a^x \frac{d}{dx} (\underbrace{x \log a}_{= \log a^x}) = a^x \cdot \log a.$$

Even easier

$$\frac{d}{dx} e^x = e^x \quad \text{since } \log e = 1$$