More examples

$$\frac{d}{dx} e^u = \frac{d}{du} e^u \cdot \frac{du}{dx} = e^u \cdot \frac{du}{dx}.$$

$$\int x e^{-x^2} dx = -\frac{1}{2} \int e^u du =$$

$u = -x^2 \qquad du = -2x\,dx$

$x\,dx = -\frac{1}{2} du$

$$-\frac{1}{2} e^u + C = -\frac{1}{2} e^{-x^2} + C$$

Warning There is no simple formula for $\int e^{-x^2} dx$.

Similar in nature to the corresponding statement about $\sin x^2$.

Limit problems

Show $\lim_{x \to \infty} x^k e^{-x} = 0$

Use l'Hospital's Rule:

$$\lim_{x \to \infty} \frac{x^k}{e^x} = \lim_{x \to \infty} \frac{k x^{k-1}}{e^x} =$$

$$\lim_{x \to \infty} \frac{k(k-1) x^{k-2}}{e^x} = \dots = \lim_{x \to \infty} \frac{k!}{e^x} = 0.$$

Equivalent version

$\lim_{x \to \infty} \frac{e^x}{x^k} = +\infty$ e^x grows faster than x^k all k.

Similarly, if $p(x)$ is any poly.

$\lim_{x \to \infty} \frac{p(x)}{e^x} = 0$ or $\lim_{x \to \infty} \frac{e^x}{p(x)} = \infty.$

<u>SLOGAN</u> Growth of exponential function much faster than growth of a polynomial function.

Finally

$u = \cos x$

$$\int \tan x\, dx = \int \frac{\sin x\, dx}{\cos x} =$$

$$\int \frac{-du}{u} = -\int \frac{du}{u} = -\log|u| + C$$

log only defined for positive variables

$$= -\log|\cos x| + C$$

$$\left(\sec = \frac{1}{\cos}\right) = \log|\sec x| + C$$

$u = \sin x$

$$\int \sec x\, dx = \int \frac{dx}{\cos x} = \int \frac{\cos x\, dx}{\cos^2 x} =$$

$$\int \frac{du}{1-u^2}.$$

ALGEBRA $\quad \dfrac{1}{1-u^2} = \dfrac{1}{2}\left(\dfrac{1}{1-u} + \dfrac{1}{1+u}\right).$

$$\int \frac{du}{1-u^2} = \frac{1}{2}\int \frac{du}{1-u} + \frac{1}{2}\int \frac{du}{1+u} =$$

$v = 1-u \qquad w = 1+u$

$$-\frac{1}{2}\int \frac{dv}{v} + \frac{1}{2}\int \frac{dw}{w} =$$

$$\frac{1}{2}\left(\log|w| - \log|v|\right) + C =$$

$$\frac{1}{2}\left(\log|1+u| - \log|1-u|\right) + C =$$

$$\frac{1}{2}\log\left|\frac{1+\sin x}{1-\sin x}\right| + C$$

This can be taken further, but stop here.