Computing e

Not directly tied to §9.4 in the text

By definition, e is characterized by the identity $\int_1^e \frac{dx}{x} = 1$, and we know $1 < e < 4$. How do we improve this?

THEOREM. $e = \lim_{x \to 0^+} (1+x)^{1/x}$

L'Hôpital's Rule

Derivation. Recall l'Hospital's Rule:

[L'Hospital bought the naming rights!]

Suppose $\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = 0$

Then $\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}$ [= etc.]

f, g "reasonable"

CIRCULAR LOGIC WARNING

Attempt to prove $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

By l'Hospital's Rule, $\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1$.

WHAT'S WRONG??

The standard derivation of $\frac{d}{dx} \sin x = \cos x$ uses the fact that $\lim_{x \to 0} \frac{\sin x}{x} = 1$ at one step of the argument, so one is effectively using $\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$ and other input to prove that $\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1.}$

(Used this already to show

$\lim_{x \to \infty} x^k e^{-x} = 0$ for all k).

Combine with

$$\log\left(\lim_{x \to a^+} h(x)\right) = \lim_{x \to a^+} \log h(x)$$

one exists $\Longleftrightarrow$ the other does in which case they are EQUAL

if $h(x) > 0$ everywhere.

Compute $\log \lim_{x \to 0^+} (1+x)^{1/x} \approx$

$$\lim_{x \to 0^+} \log (1+x)^{1/x} = \lim_{x \to 0^+} \frac{\log(1+x)}{x} \xrightarrow[\text{rule}]{\text{l'Hospital's}}$$

$$\lim_{x \to 0^+} \frac{\frac{1}{1+x}}{1} = 1.$$ Hence

$$e = \lim_{x \to 0^+} (1+x)^{1/x}.$$

We also have $e = \lim_{x \to 0^+} (1+x)^{1/x+1}$

One way to estimate e is to look at the successive values

$$\left(1+\frac{1}{n}\right)^n, \quad \left(1+\frac{1}{n}\right)^{n+1}$$

and see what happens as n gets large.

Unfortunately, this does not work too well; after the first 20 trials we barely get an estimate which is accurate to one (!) decimal place

→ it's irrational!

$$e \approx 2.71828182845\cancel{5}9045\ldots$$

In 9C one gets a very good method for approximating e:

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right)$$

e is related to instantaneously compounded interest:

If we invest 1 dollar at R% annual rate, compounded every instant (limit of compounding daily, every hour, minute, second, ..., microsecond, ...) then after 1 year the value will be $e^{R/100}$. See compounding.pdf for details.