Inverse Trigonometric Functions

$$\begin{cases} y = \sin x & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ x = \arcsin y & -1 \le y \le 1 \end{cases}$$

$$\begin{cases} y = \cos x & 0 \le x \le \pi \\ x = \arccos y & -1 \le y \le 1 \end{cases}$$

$$\begin{cases} y = \tan x & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ x = \arctan y & -\infty < y < \infty \end{cases}$$

$$\begin{cases} y = \sec x & 0 \le x < \frac{\pi}{2} \\ & \frac{\pi}{2} < x \le \pi. \\ x = \operatorname{arcsec} y & 1 \le y < \infty \\ & -\infty < y \le -1. \end{cases}$$

TWO PIECES SINCE $\sec\frac{\pi}{2}$ is not defined

Also have arccot, arccsc. but won't need them.

Differentiation of inverse trigonometric functions

RECALL $g = f^{-1} \Rightarrow g'(y) = \dfrac{1}{f'(g(y))} = \dfrac{1}{f''(x)}$

if $x = g(y)$

APPLY $f(x) = \sin x$, so $f'(x) = \cos x$.

$g(y) = \arcsin y$ is the inverse

$$\frac{d}{dx} \arcsin y = \frac{1}{f'(\arcsin y)} = \frac{1}{\cos(\arcsin y)}.$$

What is $\cos\theta$ if $\theta = \arcsin y$?

$\sin\theta = y$

$$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-y^2}$$

Hence

$$\frac{d}{dy} \arcsin y = \frac{1}{\sqrt{1-y^2}}.$$

Now suppose $f(x) = \cos x$, so
$f'(x) = -\sin x$.

$$\frac{d}{dy} \arccos y = \frac{1}{f'(\arccos y)} =$$

$$\frac{1}{-\sin(\arccos y)};$$ This time we want to know what $\sin\theta$ is if $\cos\theta = y$.
$0 \leq \theta \leq \pi$.

Once again, $\sin\theta = \sqrt{1-\cos^2\theta}$,

so $$\frac{d}{dy} \arccos y = -\frac{1}{\sqrt{1-y^2}}$$

$\uparrow$

note the sign!

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INTEGRAL VERSION $$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$$

Now look at $f(x) = \cancel{} \tan x$

$$f'(x) = \sec^2 x$$

$$\frac{d}{dy} \arctan y = \frac{1}{\sec^2(\arctan y)}$$

Find $\sec^2\theta$ if $y = \tan\theta$.

<u>EASY</u>: $1 + \tan^2\theta = \sec^2\theta$, so

$1 + y^2 = \sec^2\theta$ and

$$\frac{d}{dy} \arctan y = \frac{1}{1+y^2}$$

<u>Reverse</u> $\displaystyle\int \frac{dx}{1+x^2} = \arctan x + C$

Finally, look at $f(x) = \sec x$

$$f'(x) = \sec x \tan x$$

$$\frac{d}{dy} \operatorname{arcsec} y = \frac{1}{\sec(\operatorname{arcsec} y)\tan(\operatorname{arcsec} y)}$$

Now sec(arc sec y), and if $\theta = \operatorname{arc sec} y$ (so $y = \sec\theta$), then $\tan\theta = \pm\sqrt{y^2-1}$. Which sign?

$y \geq 1 \iff 0 \leq \theta < \frac{\pi}{2}$ $\quad \tan\theta \geq 0$ $\quad$ + sign

$y \leq -1 \iff \frac{\pi}{2} < \theta \leq \pi$ $\quad \tan\theta \leq 0.$ $\quad$ − sign

So
$$\frac{d}{dy} \operatorname{arc sec} y = \frac{1}{y \cdot \operatorname{sgn}(y)\sqrt{y^2-1}}$$
$$= \frac{1}{|y|\sqrt{y^2-1}}$$

(fortunately, this formula is less important than the previous ones).

Examples

$\boxed{1}$ $$\int \frac{dx}{\sqrt{4-x^2}} \overset{2u=x}{=\!=} \int \frac{2\,du}{\sqrt{4-4u^2}} =$$

$$\int \frac{2\,du}{2\sqrt{1-u^2}} = \arcsin u + C =$$

$$\arcsin \frac{x}{2} + C.$$

$\boxed{2}$ $$\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dx}{\sqrt{1-1+2x-x^2}} =$$

$$\int \frac{dx}{\sqrt{1-(x-1)^2}} \underset{x-1}{\overset{u=}{=\!=}} \int \frac{du}{\sqrt{1-u^2}} =$$

$$\arcsin u + C = \arcsin(x-1) + C.$$

$\boxed{3}$ $$\int \frac{dx}{x^2+2x+2} = \int \frac{dx}{(x+1)^2+1}$$

$$\stackrel{u=x+1}{=\!=} \int \frac{du}{u^2+1} = \arctan u + C =$$

$$\arctan(x+1) + C.$$

$\boxed{4}$ $$\int \frac{dx}{x^2+9} \stackrel{x=3u}{=\!=} \int \frac{3du}{9u^2+9} =$$

$$\frac{1}{3}\int \frac{du}{u^2+1} = \frac{1}{3}\arctan u + C =$$

$$\frac{1}{3}\arctan \frac{x}{3} + C.$$