More uses of trigonometric identities

Compute

$$\int_0^{2\pi} \sin mx \cos nx \, dx$$

$$\left.\begin{aligned} &\int_0^{2\pi} \sin mx \sin nx \, dx \qquad m \neq n \\ &\int_0^{2\pi} \cos mx \cos nx \, dx \qquad m \neq n \end{aligned}\right]$$

Cases $m = n$ essentially in notes1001.pdf

Ideas Use identities for $\sin(\alpha+\beta)$, $\cos(\alpha+\beta)$: They yield

$$\sin\alpha \cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right)$$

$$\sin\alpha \sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right)$$

$$\cos\alpha \cos\beta = \frac{1}{2}\left(\cos(\alpha+\beta) + \cos(\alpha-\beta)\right).$$

Details: simplify right hand expressions

$$\frac{1}{2}\left(\sin(\alpha+\beta)+\sin(\alpha-\beta)\right)=\frac{1}{2}\left(\sin\alpha\cos\beta+\cancel{\cos\alpha\sin\beta}+\sin\alpha\cos\beta-\cancel{\cos\alpha\sin\beta}\right)$$

$$\begin{aligned}\frac{1}{2}\left(\cos(\alpha-\beta)-\cos(\alpha+\beta)\right)&=\frac{1}{2}\left(\cancel{\cos\alpha\cos\beta}+\sin\alpha\sin\beta-(\cancel{\cos\alpha\cos\beta}-\sin\alpha\sin\beta)\right)\\&=\frac{1}{2}\left(\sin\alpha\sin\beta-(-\sin\alpha\sin\beta)\right)\\&=\frac{1}{2}\left(2\sin\alpha\sin\beta\right).\end{aligned}$$

$$\frac{1}{2}\left(\cos(\alpha+\beta)+\cos(\alpha-\beta)\right)=\frac{1}{2}\left(\cos\alpha\cos\beta-\cancel{\sin\alpha\sin\beta}+\cos\alpha\cos\beta+\cancel{\sin\alpha\sin\beta}\right)$$

[—— denotes canceling terms].

Apply these identities to evaluate the integrals:

$$\sin mx \cos nx = \frac{1}{2}\left(\sin(m+n)x + \sin(m-n)x\right).$$

Do the case where $m = n$ first. Then

$$\sin mx \cos mx = \frac{1}{2} \sin 2mx, \text{ and}$$

$$\int_0^{2\pi} \sin mx \cos mx \, dx = \frac{1}{2}\int_0^{2\pi} \sin 2mx \, dx = -\frac{\cos 2mx}{4m}\Bigg|_0^{2\pi}$$

$$= 0.$$

Now suppose $m \neq n$.

Then we have

$$\int_0^{2\pi} \sin mx \cos nx \, dx = \frac{1}{2}\int_0^{2\pi} \left(\sin(m+n)x + \sin(m-n)x\right) dx =$$

$$\frac{1}{2}\left(-\frac{\cos(m+n)x}{m+n} - \frac{\cos(m-n)x}{m-n}\right)\Bigg|_0^{2\pi}$$

need $m \neq n$ to write the second term as given.

$$\left[\begin{array}{l} \cos 2(m+n)\pi = \cos 0 = 1 \\ \cos 2(m-n)\pi = \cos 0 = 1 \end{array}\right]$$

$= 0$ once again.

Next,

$$\sin mx \sin nx = \frac{1}{2}\left(\cos(m-n)x - \cos(m+n)x\right)$$

$$\int_0^{2\pi} \sin mx \sin nx \, dx = \frac{1}{2}\int_0^{2\pi}\left(\cos(m-n)x - \cos(m+n)x\right)dx =$$

$$\frac{1}{2}\left(\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right)\Bigg|_0^{2\pi} = 0.$$

$$\left[\begin{array}{l} \sin 2(m-n)\pi = \sin 0 = 0 \\ \sin 2(m+n)\pi = \sin 0 = 0 \end{array}\right]$$

A similar use of

$$\cos mx \cos nx = \frac{1}{2}\left(\cos(m+n)x + \cos(m-n)x\right)$$

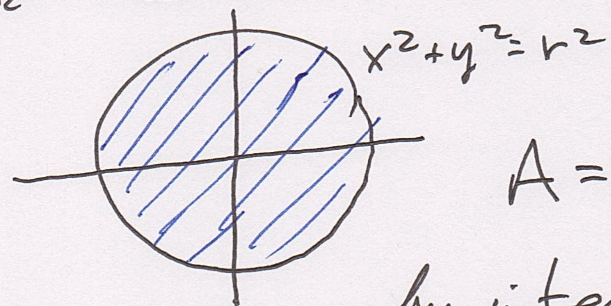
shows that

$$\int_0^{2\pi} \cos mx \cos nx \, dx = 0.$$

Trigonometric Substitutions

Motivating problem

Check the area formula



$A = \pi r^2$

by integral calculus.

Should have $A = 4\int_0^r \sqrt{r^2-x^2}\,dx$.

So we want to find $\int \sqrt{r^2-x^2}\,dx$ and

check that it yields $\int_0^r \sqrt{r^2-x^2}\,dx = \frac{\pi r^2}{4}$.

Start with case $r=1$

$\int \sqrt{1-x^2}\,dx = ??$

Let $x = \sin\theta$, change variables.

$$\int \sqrt{1-x^2}\, dx = \int \cos\theta \, d\sin\theta =$$

$x = \sin\theta \Rightarrow$

$\theta = \arcsin x$

$$\int \cos^2\theta \, d\theta \overset{\text{last}}{\underset{\text{time}}{=}}$$

$$\frac{\theta}{2} - \frac{\sin 2\theta}{4} + C.$$

To write $\sin 2\theta$ in terms of x, use

$$\sin 2\theta = 2\sin\theta\cos\theta = 2x\underbrace{\sqrt{1-x^2}}$$

if $\sin\theta = x$, then $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}$

Thus we get

$$\int \sqrt{1-x^2}\, dx = \frac{\arcsin x}{2} + \frac{x\sqrt{1-x^2}}{2} + C.$$

and $$\int_0^1 \sqrt{1-x^2}\, dx = \frac{\arcsin x}{2} + \frac{x\sqrt{1-x^2}}{2}\Bigg|_0^1 = \frac{1}{2}\frac{\pi}{2} = \frac{\pi}{4}.$$

zero at 1 + 0

More generally, find $\int \sqrt{r^2-x^2}\, dx$.

Change variables $x = ru$

So $\sqrt{r^2-x^2} = r\sqrt{1-u^2}$ and

$dx = r\,du$

$$\int \sqrt{r^2-u^2}\,du = r^2 \int \sqrt{1-u^2}\,du =$$

$$r^2\left(\frac{\arcsin u}{2} + \frac{u\sqrt{1-u^2}}{2}\right) + C =$$

$$\frac{r^2}{2}\arcsin\frac{x}{r} + \frac{ru \cdot r\sqrt{1-u^2}}{2} + C =$$

$$\frac{r^2}{2}\arcsin\frac{x}{r} + \frac{x \cdot \sqrt{r^2-x^2}}{2} + C.$$

Hence $\frac{A}{4} = \int_0^r \sqrt{r^2-x^2}\,dx =$

$$\frac{r^2}{2}\arcsin\frac{x}{r}\Big|_0^r + \frac{1}{2}x\sqrt{r^2-x^2}\Big|_0^r = \frac{\pi r^2}{4}$$

$\boxed{\text{what we expected}}$

$\left(\frac{\pi}{2} - 0\right)$

zero when $x = 0, r$