Integration by Parts

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$ translates to

$$\int d(uv) = \int u\,dv + \int v\,du$$ or

$$\int u\,dv = u\cdot v - \int v\,du.$$

Typical examples

$$\int \underbrace{x}_{u}\cdot\underbrace{e^x dx}_{dv} = \overset{u}{x}\cdot\overset{v}{e^x} - \int \overset{v}{e^x}\cdot\overset{du}{dx}$$

$u=x,\ v=e^x$

$$= xe^x - e^x + C$$

$$\int \log x\,dx = \overset{u}{x}\,\overset{v}{\log x} - \int \overset{v}{x}\cdot \overset{du}{d(\log x)} =$$

$u=\log x$
$v=x$

$$x\log x - \int x\cdot\frac{dx}{x} =$$

$$x\log x - \int dx =$$

$$x\log x - x + C.$$

$$\int x^2 e^x\,dx = x\cdot v - \int v\cdot dx =$$

$x = u$, $xe^x dx = dv$

$$x\,(xe^x - e^x) - \int (xe^x - e^x)\,dx.$$

So we can find a closed formula for $\int x^2 e^x\,dx$. Likewise, can find a closed formula for $\int x^n e^x\,dx$ recursively.

Warning: No good formula for

$$\int \frac{e^x - 1}{x}\,dx.$$

Likewise for $\frac{\sin x}{x}$ [value is 0 at $x=0$]

Partial Fractions

Want to evaluate $\int \frac{p(x)\,dx}{q(x)}$, where $\deg p(x) < \deg q(x)$.

Write $\frac{p(x)}{q(x)}$ as a sum of pieces

$$\frac{C}{(x-a)^k}, \quad \frac{E}{[(x-p)^2+q^2]^k}, \quad \frac{Fx}{[(x-p)^2+q^2]^k}$$

quadratic, no realroots

need $q \neq 0$

where the denominators divide $q(x)$

For example,

$$\frac{p(x)}{(x-1)^3} = \frac{C_1}{(x-1)} + \frac{C_2}{(x-1)^2} + \frac{C_3}{(x-1)^3}.$$

Example Find $\int \frac{dx}{x^3-x}$

Write $\frac{1}{x^3-x} = \frac{a}{x-1} + \frac{b}{x} + \frac{c}{x+1} =$

$$\frac{a(x^2+x)}{x^3-x} + \frac{b(x^2-1)}{x^3-x} + \frac{c(x^2-x)}{x^3-x} =$$

$$\frac{\overset{=0}{(a+b+c)}x^2 + \overset{=0}{(a-c)}x + \overset{=1}{(-b)}}{x^3-x}$$

set coefficients equal to those in original numerator.

Solve for a, b, c $\quad b=-1$

$a-c=0 \Rightarrow a=c$

$a+b+c=0 \quad 2a-1=0$

$\Rightarrow a=c=\frac{1}{2}.$

Hence $\frac{1}{x^3-x} = \frac{1}{2}\cdot\frac{1}{x+1} - \frac{1}{x} + \frac{1}{2}\frac{1}{x-1}$ and

$$\int\frac{dx}{x^3-x} = \frac{1}{2}\int\frac{dx}{x+1} - \int\frac{dx}{x} + \frac{1}{2}\int\frac{dx}{x-1} =$$

$$\frac{1}{2}\log(x+1) - \log x + \frac{1}{2}\log(x-1) + C.$$

<u>Question</u> Suppose the denominator is $q(x)$ $(x^2+1)^2(x^2-1)^2$. What sort of pieces does one need to expand $\frac{h(x)}{q(x)}$? $\deg h < 8$.

Constants times the following: $(x^2+1)^2(x-1)^2(x+1)^2$

$$\frac{1}{(x-1)},\ \frac{1}{(x-1)^2},\ \frac{1}{(x+1)},\ \frac{1}{(x+1)^2}$$
$$\frac{1}{(x^2+1)},\ \frac{x}{(x^2+1)},\ \frac{1}{(x^2+1)^2},\ \frac{x}{(x^2+1)^2}.$$

The number of terms = degree of denominator.

<u>RECALL</u> x^2+bx+c has no real roots $\Longleftrightarrow$ $b^2<4c$. In this case we can write it by completing the square:

$$x^2+bx+c = \left(x+\frac{b}{2}\right)^2 + \left(c-\frac{b^2}{4}\right) =$$

$(x-p)^2 + q^2 \qquad p = -\frac{b}{2}$

$q = \sqrt{c - \frac{b^2}{4}}$ ← positive because $4c > b^2 \Rightarrow c - \frac{b^2}{4} > 0$.

Another example

Find $\displaystyle\int \frac{3x+2}{(x-1)^2}\,dx$.

The first step is to write the integrand as $\dfrac{a}{(x-1)} + \dfrac{b}{(x-1)^2}$ where a & b are to be determined.

Then $\dfrac{a}{(x-1)} + \dfrac{b}{(x-1)^2} = \dfrac{a(x-1)}{(x-1)^2} + \dfrac{b}{(x-1)^2} =$

$\dfrac{ax+(b-a)}{(x-1)^2}$. We want to find the right values of a and b such that this equals $\dfrac{3x+2}{(x-1)^2}$. Set the coeffs of x & 1 equal:

$a = 3$, $b - a = 2$

So $b - 3 = 2$ and hence $b = 5$, and we have

$$\frac{3x+2}{(x-1)^2} = \frac{3}{(x-1)} + \frac{5}{(x-1)^2}.$$

Therefore

$$\int \frac{3x+2}{(x-1)^2}\,dx = 3\int \frac{dx}{(x-1)} + 5\int \frac{dx}{(x-1)^2} \underset{u=x-1}{=}$$

$$3\int \frac{du}{u} + 5\int \frac{du}{u^2} = 3\log u - \frac{5}{u} + C =$$

$$3\log(x-1) - \frac{5}{(x-1)} + C.$$