Population Growth

Unrestrained growth

$P(t) = P_0 e^{kt}$. $P' = kP$

Good model for growth at early stages, but unsustainable in the long run. Growth must slow down and level off.

Logistic Equation

$$y' = ay(1-cy) =$$
$$ay - acy^2$$ c = small positive constant.

PROBLEM Solve this equation

$$\frac{y'}{ay - acy^2} = 1 \implies$$

$$\int \frac{dy}{ay - acy^2} = \int 1\,dx = x + K$$ K constant.

Use partial fractions to describe L.H.S.

$$\frac{1}{y-cy^2} = \frac{1}{y(1-cy)} = \frac{V}{y} + \frac{W}{1-cy}$$

$$\text{R.H.S.} = \frac{V(1-cy)+Wy}{y(1-cy)} = \frac{\overset{=1}{V}+y\overset{=0}{(W-cV)}}{y(1-cy)}$$

So $W=c$

$$\boxed{\frac{1}{y-cy^2} = \frac{1}{y} + \frac{c}{1-cy}}$$ and hence

$$\frac{1}{a}\int\frac{dy}{y} + \frac{c}{a}\int\frac{dy}{1-cy} = x+K$$

$$= \frac{1}{a}\log y - \frac{1}{a}\log(1-cy) = \frac{1}{a}\log\left(\frac{y}{1-cy}\right)$$

$$\boxed{\begin{aligned}&\int\frac{dy}{1-cy} \underset{z=1-cy}{=} \\ &\int\frac{-\frac{1}{c}dz}{z} = \\ &-\frac{1}{c}\log z = \\ &\quad -\frac{1}{c}\log(1-cy)\end{aligned}}$$

EXPONENTIATE

$$L_0 e^x = \left(\frac{y}{1-cy}\right)^{1/a} \Longrightarrow$$

$$Le^{ax} = \frac{y}{1-cy}.$$

$$Le^{ax}(1-cy) = y \Longrightarrow$$

$$Le^{ax} = y + Le^{ax}cy$$

$$y = \frac{Le^{ax}}{1+cLe^{ax}}.$$ [recall that c is pretty small]

This is close to Le^{ax} if x is close to zero, but

$$\lim_{x\to\infty} \frac{Le^{ax}}{1+cLe^{ax}} = \lim_{x\to\infty} \frac{1}{c+L^{-1}e^{-ax}}$$

divide num. + denom. by Le^{ax}

$$= \frac{1}{c}.$$

maybe large, but bounded

Alternate form of solution

$$y = \frac{1}{c+Me^{-ax}}$$