Summarizing integration techniques

① No absolute guarantees that a specific approach will work, but there are some useful guidelines.

② Computations can get long and messy.

[Goal to minimize for exam problems.]

CHANGE OF VARIABLES

Recognize that the integral has a form close to

$$\int f(u)\,du = \int f(u(x))\,\frac{du}{dx}\,dx .$$

Ex. $\displaystyle\int \frac{x\,dx}{(x^2+1)^2} = \frac{1}{2}\int \frac{du}{u^2}$ $\quad u = x^2+1$.

TRIGONOMETRIC IDENTITIES / SUBSTITUTIONS

Can integrate $\int \sin^k x\,dx$, $\int \cos^k x\,dx$.

via trig identities.

(best cases $k = 2, 3, 4$).

Guidelines for some changes of variables

If you see $\left\{\begin{matrix} \sqrt{a^2-x^2} \\ \sqrt{a^2+x^2} \\ \sqrt{x^2-a^2} \end{matrix}\right\}$ try $\left\{\begin{matrix} x = a\sin\theta \\ x = a\tan\theta \\ x = a\sec\theta \end{matrix}\right\}$.

Then the radical expression becomes

$\left\{\begin{matrix} a\cos\theta \\ a\sec\theta \\ a\tan\theta \end{matrix}\right\}$.

Examples: $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}}$

$\displaystyle\int \frac{dx}{x^2+a^2}$

Integration by parts

$$\int u\,dv = uv - \int v\,du$$

Searching for the right choice of u s.t. du is a real simplifying factor.

L logarithmic functions
I inverse trig functions
A algebraic functions [for example $u = x$]
T trig functions
E exponential functions.

$\int \log x\,dx$, $\int \arcsin x\,dx =$

$x \arcsin x - \int \frac{x\,dx}{\sqrt{1-x^2}}$

now use change of variables

$\int x e^x\,dx$

More detailed computation

$$\int \underset{u}{\arcsin x} \cdot \underset{\substack{dv \\ v=x}}{dx} = \underset{v \cdot u}{x \cdot \arcsin x} - \int \underset{v}{x} \, \frac{dx}{\underset{du}{\sqrt{1-x^2}}} =$$

$$x \cdot \arcsin x - \int \frac{x\,dx}{\sqrt{1-x^2}} =$$

let $w = 1-x^2$

$x\,dx = -\frac{1}{2}\,dw$

$$x \cdot \arcsin x + \frac{1}{2}\int \frac{dw}{\sqrt{w}} =$$

$$x \cdot \arcsin x + \frac{1}{2}\,\frac{w^{1/2}}{\frac{1}{2}} + C =$$

$$x \cdot \arcsin x + \sqrt{w} + C =$$

$$x \cdot \arcsin x + \sqrt{1-x^2} + C.$$

One can (should!) check this by taking the derivative of the result + verifying it simplifies to $\arcsin x$.

$$\frac{d}{dx}\left(x \cdot \arcsin x + \sqrt{1-x^2}\right) =$$

$$\arcsin x + \frac{x}{\sqrt{1-x^2}} + \frac{1}{2}(1-x^2)^{-1/2}(-2x) =$$

$$\arcsin x + \cancel{\frac{x}{\sqrt{1-x^2}}} + \cancel{\frac{1}{2}\cdot(-2)\frac{x}{\sqrt{1-x^2}}}$$

$= \arcsin x$, so the answer is OK!!

Followup problem:

Find $\int \arctan x \, dx$ using integration by parts:

$$u = \arctan x$$

$$v = x$$

Partial fractions

Split the expression $\frac{p(x)}{q(x)}$ POLYNOMIALS

into more manageable pieces using algebra.

First step: $\frac{p(x)}{q(x)} = a(x) + \frac{b(x)}{q(x)}$ ← polynomial with degree$(b)<$ degree(q).

Polynomial

Second step. $\frac{b(x)}{q(x)}$ is a sum of pieces

$\frac{b_i(x)}{q_i(x)}$ for some polys $q_i(x)$ dividing $q(x)$

Namely the q_i's are powers of either

$\boxed{A}$ linear polynomials

$\boxed{B}$ quadratic polynomials with no real roots.

If $q_i(x) = (x-c)^m$, take $b_i(x)$ to be constant.

If $q_i(x) =$ (quad., no real roots)^m, take $b_i(x)$ to be a linear polynomial.

Examples $\boxed{1}$ $\dfrac{x^3+x^2+1}{(x-1)^4} =$

$$\frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{(x-1)^4} =$$

$$\frac{D + C(x-1) + B(x-1)^2 + A(x-1)^3}{(x-1)^4}$$

The numerator is a 3rd degree polynomi x, and it must be equal to x^3+x^2+1. Equate the coefficients of $1, x, x^2, x^3$ in both expressions, obtaining 4 eqns. in A, B, C, D that we can solve for these coefficients.

$\boxed{2}$ $\dfrac{x^3+2x^2+4x-8}{(x-1)^2(x^2+4)} =$

$$\frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+4}.$$

In this case we may rewrite the right hand side as

$$\frac{A(x-1)(x^2+4) + B(x^2+4) + (Cx+D)(x-1)^2}{(x-1)^2(x^2+4)}.$$

$\boxed{3}$

$$\frac{x+3}{x^2+4x+5} = \frac{x+3}{(x+2)^2+1} \overset{u=x+2}{=\!=\!=} \frac{u+1}{u^2+1}.$$

So $\displaystyle\int \frac{(x+3)\,dx}{x^2+4x+5} =$

$$\int \frac{(u+1)\,du}{u^2+1} = \underbrace{\int \frac{u\,du}{u^2+1}}_{v=u^2+1} + \int \frac{du}{u^2+1} =$$

$$\frac{1}{2}\int \frac{dv}{v} + \int \frac{du}{u^2+1} = \frac{1}{2}\log v + \arctan u + C$$

$$= \frac{1}{2}\log(u^2+1) + \arctan u + C =$$

$$\frac{1}{2}\log(x^2+4x+5) + \arctan(x+2) + C.$$

In practice, first year calculus exercises/exam problems rarely have factors like $(x^2+9)^2$, $(x^2+6x+25)^4$ [irreducible quadratic to a power > 1].

Good examples to compare:

$$\int \frac{dx}{x^2+2x-3}, \quad \int \frac{x\,dx}{x^2+2x-3}$$

$$\int \frac{dx}{x^2+2x+1}, \quad \int \frac{x\,dx}{x^2+2x+1}$$

$$\int \frac{dx}{x^2+2x+5}, \quad \int \frac{x\,dx}{x^2+2x+5}$$

Note that the denominators can be rewritten as

$$(x+3)(x-1)$$
$$(x+1)^2$$
$$(x+1)^2+2^2.$$

Notwithstanding the remark on the top of p. 5, the following should be understood, and similar questions might appear on exams: [odds ~ 2 out of 3]

Suppose we have $\frac{b(x)}{q(x)}$ where

$$q(x) = (x-1)(x+2)^2(x^2+4)^3.$$

$\deg b(x) < 9 = \deg q(x)$

Describe the terms in the partial fraction expansion of $\frac{b(x)}{q(x)}$ with coefficients T.B.D.

Solution There are $\boxed{9 = \text{degree } q(x)}$ terms:

$$\frac{A}{(x-1)},\ \frac{B}{(x-2)},\ \frac{C}{(x-2)^2},\ \frac{Dx}{x^2+4},\ \frac{E}{x^2+4},$$

$$\frac{Fx}{(x^2+4)^2},\ \frac{G}{(x^2+4)^2},\ \frac{Hx}{(x^2+4)^3},\ \frac{K}{(x^2+4)^3}$$