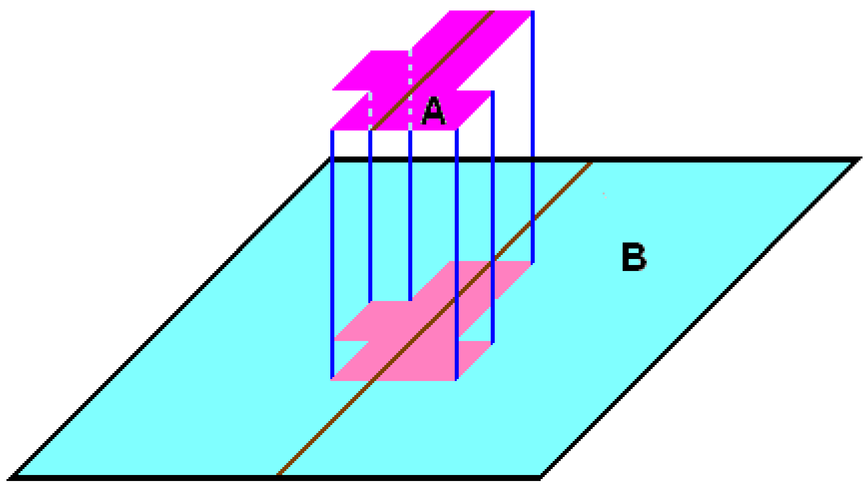
Centroids and moments

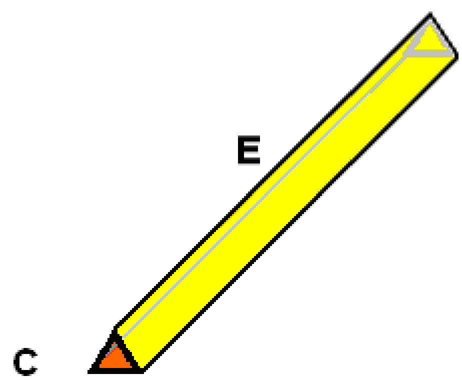
The purpose of this note is to provide some physical motivation for the standard formulas which give the center of mass of an object in the plane. Suppose **A** is a bounded planar object whose center of mass (or centroid) has coordinates (x^*, y^*), and choose positive numbers a and b such that all points on **A** belong to the solid rectangular region **B** defined by the following inequalities:

$$x^* - a \le x \le x^* + a$$
$$y^* - b \le y \le y^* + b$$

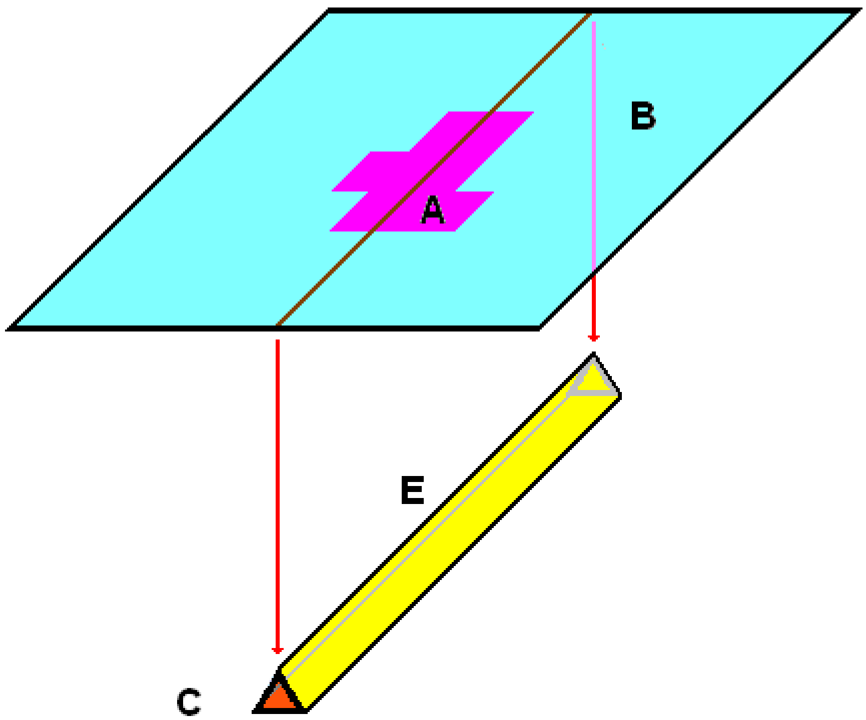
Think of **B** as a flat, firm rectangular sheet with uniform density (made of glass, metal, wood, plastic, *etc.*) such that **A** rests on top of **B**.



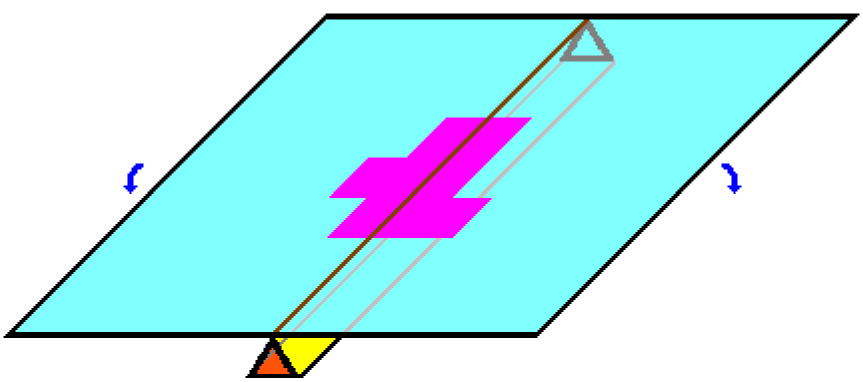
Next, suppose that we have a triangular rod **C** with equilateral ends, positioned so that one of the lateral faces is horizontal and the opposite edge **E** lies above this face.



As suggested by the figure on the next page, suppose that we now we rest **B** and **A** on the edge **E** of **C** along the vertical line defined by the equation $x = x^*$.

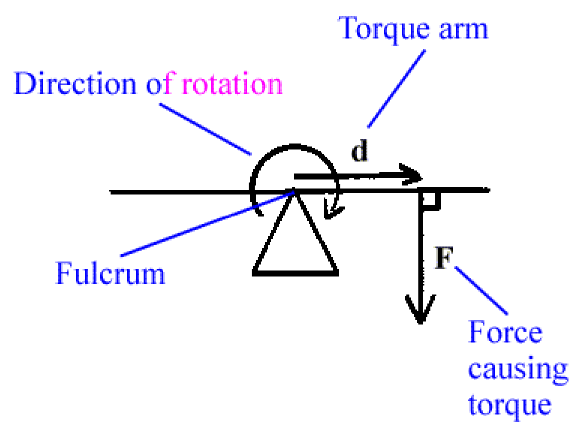


Since we are resting **A** and **B** along a line containing the center of mass for this combined physical system of objects, ***we expect that the combined object will balance perfectly***, not tipping either to the left or right.



Physically, this means that the total ***torque*** (pronounced "TORK") or ***moment*** of **A** to the ***right*** of the vertical line defined by the equation $x = x^*$ is ***equal*** to the total torque or moment of **A** to the ***left*** of the vertical line defined by the equation $x = x^*$.

Before proceeding, we need to define torque precisely in the simplest case. Suppose that we are given a point mass P of g units in the coordinate plane at the point (x, y). Then the torques or moments of P with respect to the $x-$ and $y-$ axes are equal to the products gy and gx respectively. A positive torque with respect to the $x-$ axis indicates that the torque will tilt the plane in the clockwise direction, while a negative torque indicates that the torque will tilt the plane in the counterclockwise direction. Similar considerations hold if one considers the $y-$ axis. In the drawing below, the fulcrum is assumed to support the plane along the $y -$ axis, the distance of the point mass from the axis is denoted by d, and $\mathbf{F}$ represents the force of gravity on the point mass. For this example, the torque is equal to $\mathbf{F}$ times d.

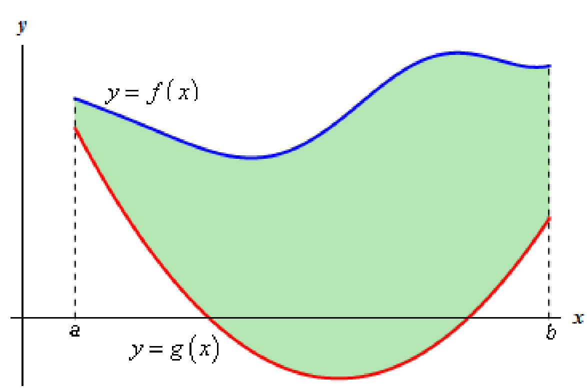


(***Source:*** **http://zonalandeducation.com/mstm/physics/mechanics/forces/torque/torque2.gif**)

Let's assume now that the object represented by **A** has uniform density, which we shall take to be 1 for the sake of convenience, and let's temporarily restrict attention to regions defined by inequalities of the form

$$a \leq x \leq b, \qquad g(x) \leq y \leq f(x)$$

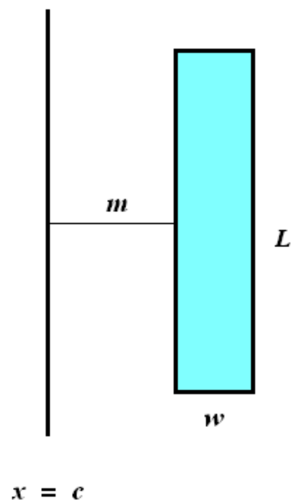
as in the picture on the next page. Our goal will be to *find the* $x -$ *coordinate* x^* *of the center of mass for* **A.**



(Source: http://tutorial.math.lamar.edu/Classes/CalcII/CenterOfMass.aspx)

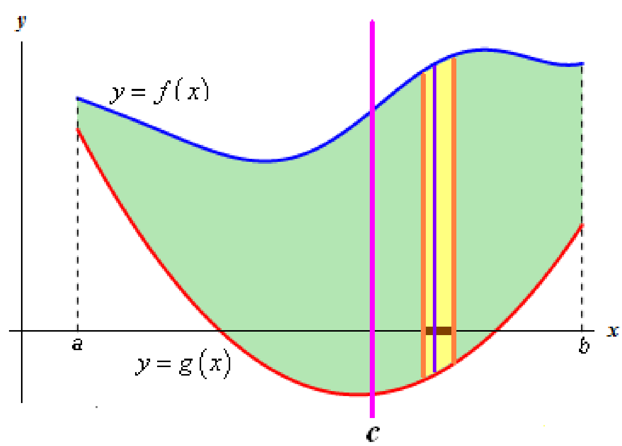
Since a and b are the smallest and largest coordinates for the points in **A,** it should be clear that the coordinate x^* of the center point should be somewhere between these two extremes.

Let's look more generally at the following problem: Suppose that the region is lying on a flat plate as in the discussion of pages 1 and 2, and this flat plate is supported by a triangular wedge whose top edge coincides with the vertical line $x = c$. How do we measure the total torque to the right and left of that line? As in earlier discussions, the key to doing this is to approximate the region by thin nonoverlapping rectangular strips and to estimate the torque on each strip. In order to do this, we need to get good estimates for the torque on a rectangular strip. More precisely, we would like to get upper and lower estimates which are reasonably close if the width of the strip is sufficiently small. We shall use the drawing below as motivation.



Since we are assuming unit density, the mass of the strip is equal to the area Lw of the strip. We can get an upper estimate for the torque by assuming that all the mass of the rectangular strip is concentrated on the right hand edge. If this were the case then the torque would be equal to $(m + w)Lw$. Likewise, we can get a lower estimate by assuming that all the mass is concentrated on the left hand edge, which would yield a torque of mLw. If the strip is sufficiently thin, then the difference between the upper and lower estimates is smaller by an order of magnitude.

Now let's apply this to previous drawing and estimate the torque of the region depicted there. The idea is to estimate the torques of thin strips like the yellow region and add them up.



By the previous discussion, the torque is approximately the product of the width, which we shall call Δx, the distance to the pink line, which equals $x - c$, and the length, which we approximate by $f(x) - g(x)$ for some choice of x in the brown interval (hence the length is approximated by the length of the purple segment). If we now add these up and take limits as the maximum width goes to zero, we find that the total torque about the vertical line $x = c$ is given by the following integral:

$$\int_a^b (x-c)\cdot(f(x)-g(x))\,dx$$

If we now stipulate that c is the $x-$coordinate x^* for the center of mass of **A,** then by the discussion at the beginning we know that the torque to the left of c will balance out the torque to the right, so that the total torque is zero and we have the following equation:

$$\int_a^b (x-x^*)\cdot(f(x)-g(x))\,dx = 0$$

This immediately yields the standard basic formula describing the $x-$coordinate for the center of mass:

$$\int_a^b x\cdot(f(x)-g(x))\,dx = x^*\cdot\int_a^b (f(x)-g(x))\,dx = x^*\cdot Area(\mathrm{A})$$

Similar considerations apply if the roles of the $x-$ and $y-$ coordinates are switched.