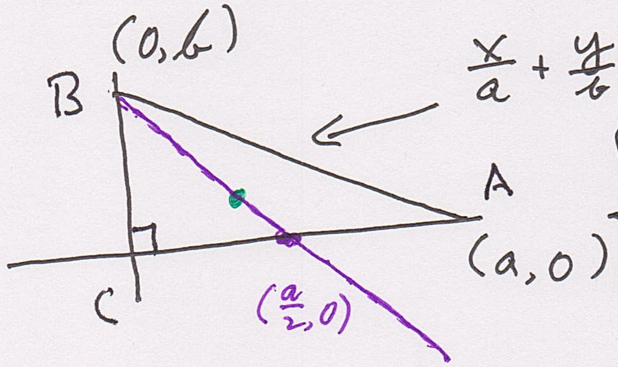
Finding centroids

(centers of mass / gravity)

Look at very familiar examples

Area of $\Delta ABC = \frac{ab}{2}$



$\frac{x}{a}+\frac{y}{b}=1$, or $\begin{cases} y = b - \frac{bx}{a}. \\ x = a - \frac{ay}{b} \end{cases}$

We expect the centroid should lie on the line joining B to the midpoint of edge [CA]

centroid coords = (x^*, y^*).

$$x^* = \frac{1}{\text{area}} \int_0^a x\left(b - \frac{bx}{a}\right)dx =$$

$$\frac{2}{ab}\int_0^a x\left(b - \frac{bx}{a}\right)dx =$$

$$\frac{2}{ab}\left(\frac{bx^2}{2}-\frac{bx^3}{3a}\right)\Bigg|_0^a =$$

$$\frac{2}{\cancel{a}\cancel{b}}\left(\frac{\cancel{b}\cancel{a^2}^{\,a}}{2}-\frac{\cancel{b}\cancel{a^3}^{\,a}}{3\cancel{a}}\right) = 2\left(\frac{a}{2}-\frac{a}{3}\right)=\frac{a}{3}.$$

Similarly, $y^* = \frac{b}{3}$ (VERIFY THIS).

Why does this lie on the line joining $(0,b)$ to $(\frac{a}{2},0)$?

The line in question has eqn

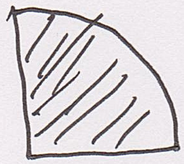
$$\frac{y}{b}+\frac{x}{a/2}=1, \text{ or } \frac{y}{b}+\frac{2x}{a}=1.$$

Now $\frac{y^*}{b}+\frac{2x^*}{a} = \frac{1}{\cancel{b}}\left(\frac{\cancel{b}}{3}\right)+\left(\frac{2}{\cancel{a}}\right)\left(\frac{\cancel{a}}{3}\right) =$

$\frac{1}{3}+\frac{2}{3}=1$, so (x^*, y^*) is on the line joining $B=(0,b)$ to $(\frac{a}{2},0)$ = midpoint of the segment $[AC]$.

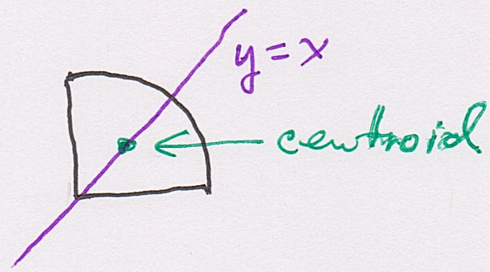
Find the centroid of the quarter circle region

$$x^2+y^2 \le 1$$
$$x, y \ge 0$$



$$0 \le x \le 1$$
$$0 \le y \le \sqrt{1-x^2}$$

By symmetry $x^* = y^*$



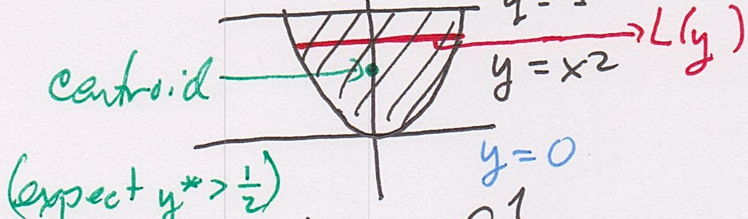
area $= \frac{\pi}{4}$. So $y^* =$

$$x^* = \frac{1}{A = \text{area}} \int_0^1 x\sqrt{1-x^2}\,dx \underset{u=1-x^2}{=\!=}$$

$$-\frac{1}{2A}\int_{u=1}^{u=0} \sqrt{u}\,du \underset{\text{FLIP INTEGRAL}}{=\!=} \frac{1}{2A}\int_0^1 u^{1/2}\,du =$$

$$\frac{1}{\cancel{2}A}\cdot\frac{\cancel{2}}{3}u^{3/2}\Big|_0^1 = \frac{4}{3\pi} \approx 0.424413182\ldots$$

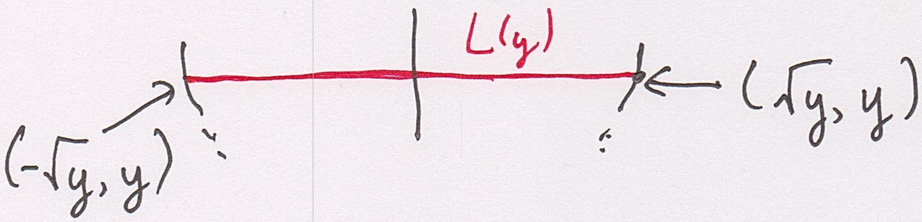
Find the centroid for the region bounded by $y = x^2$ and $y = 1$.



$x^* = 0$ by symmetry

$$y^* = \frac{1}{\text{area}} \int_0^1 y\, L(y)\, dy$$

$$\text{Area} = \int_{-1}^{1} (1 - x^2)\, dx = x - \frac{x^3}{3} \Big|_0^1 = \frac{4}{3}.$$



so $L(y) = 2\sqrt{y}$

Hence $y^* = \dfrac{1}{\frac{4}{3}} \displaystyle\int_0^1 2 y^{3/2} dy = \frac{3}{2} \int_0^1 y^{3/2} dy =$

$$\frac{3}{2} \cdot \frac{2}{5}\, y^{5/2} \Big|_0^1 = \frac{3}{5}.$$