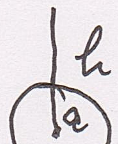
Improper integrals

Recall problem of computing energy usage for rocket



force needed at height h

$$= \frac{K}{(h+a)^2} \text{ (roughly)}$$

Energy use to lift rocket to height H =

$$\int_0^H \frac{K\,dh}{(h+a)^2} = \left.\frac{-K}{(h+a)}\right|_0^H = \frac{K}{a} - \frac{K}{(H+a)}$$

as $H \to \infty$ this has $\frac{K}{a}$ as a limit.

Convenient to write the left side as

$\displaystyle\int_0^\infty \frac{K\,dh}{(h+a)^2}$. Generally, one can ask

about $\displaystyle\lim_{b\to\infty} \int_a^b f(x)\,dx$ etc.

Say $f(x) = x^k$, say $k \neq -1$. (what if $k=-1$?)

Say $a>0$

$$\int_a^b f(x)\,dx = \int_a^b x^k\,dx =$$

$$\frac{1}{k+1} x^{k+1} \Big|_a^b = \frac{b^{k+1}}{k+1} - \frac{a^{k+1}}{k+1}.$$

Limit exists $\Leftrightarrow$ limit of first term exists.

Say $k > -1$. Then $\lim_{b \to \infty} b^{k+1} = \infty$,

so $\int_a^\infty x^k\,dx = \infty$ then.

Now say $k < -1$. Then $\lim_{b \to \infty} b^{k+1} = 0$

so $\int_a^\infty x^k\,dx = -\frac{a^{k+1}}{k+1}$. $\leftarrow$ Positive since $k+1 < 0$ $(k<-1)$.

Another type of problem What is

$$\int_a^b f(x)\,dx \text{ if } \left.\begin{matrix}\lim\limits_{x\to a^+} f(x) = \infty \\ \lim\limits_{x\to b^-} f(x) = \infty\end{matrix}\right\} ?$$

Suppose $f(x) = \dfrac{1}{x^r}$ $\quad a=0,\ b=1$

$r>0,\ r \neq 1$

Then $\displaystyle\int_0^1 \frac{dx}{x^r} = \lim_{\varepsilon\to 0} \int_\varepsilon^1 \frac{dx}{x^r} =$

$$\lim_{\varepsilon\to 0} \frac{1}{1-r} x^{1-r}\Big|_\varepsilon^1 =$$

$$\frac{1}{1-r}\lim_{\varepsilon\to 0}\left(1-\varepsilon^{1-r}\right) =$$

$$\frac{1}{1-r} - \lim_{\varepsilon\to 0}\frac{\varepsilon^{1-r}}{1-r} =$$

$$\begin{cases}\dfrac{1}{1-r} & \text{if } r<1 \\ +\infty & \text{if } r>1\end{cases}$$

$\leftarrow$ note that $\varepsilon^{1-r} \to +\infty$ and $1-r<0$

Note:
The second term is 0 if $r<1$, ∞ if $r>1$.

What if $a = 1$?

Then we have

$$\int_0^1 \frac{dx}{x} = \lim_{\varepsilon \to 0} \int_\varepsilon^1 \frac{dx}{x} = \lim_{\varepsilon \to 0} \log x \Big|_\varepsilon^1 =$$

$$\cancel{\log 1} - \lim_{\varepsilon \to 0} \log \varepsilon =$$

0

$$0 - (-\infty) = +\infty.$$

IMPORTANT Sometimes there is no ~~usable~~ meaningful way of assigning a value to $\int_a^\infty f(x)\,dx$, or to $\int_a^b g(x)\,dx$ where $\lim_{x \to a^+} g(x)$ is meaningless or not definable

[No simple examples of the second type].

$$\lim_{b\to\infty}\int_0^b \cos x\,dx = \lim_{b\to\infty} \sin x\Big|_0^b \quad \boxed{\text{NO LIMIT VALUE}}$$

One more example

Is $\displaystyle\int_0^{\pi/2} \tan x\,dx$ meaningful?

$$\lim_{b\to\pi/2}\int_0^b \tan x\,dx = \lim_{b\to\pi/2} -\log\cos x\Big|_0^b =$$

$$\lim_{b\to\pi/2} -\log\cos b = +\infty.$$

$$\left(\begin{array}{l}\cos b\to 0\\ \text{as } b\to\frac{\pi}{2}\end{array}\right)$$