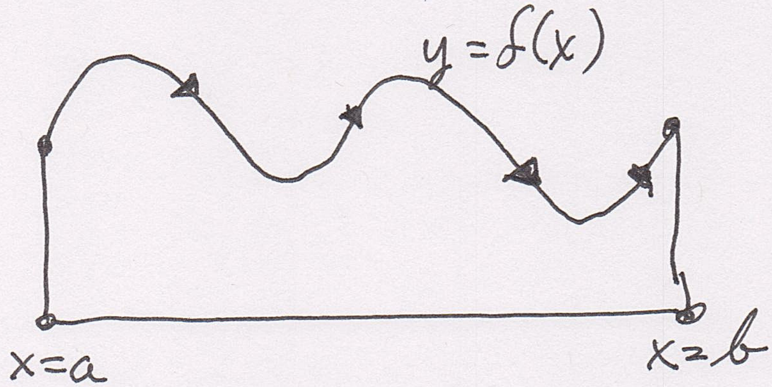
Arc length (for curves)

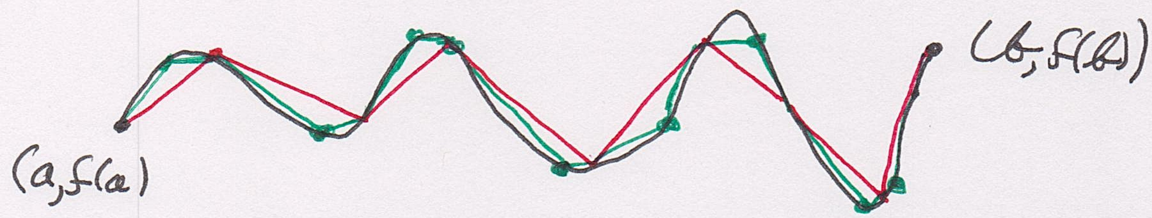
In this course only consider curves which are graphs of functions.



<u>Physical method.</u> Fit a piece of string over the curve.

<u>Disadvantage.</u> If the curve goes up and down too much, this method might not be especially accurate.

<u>Standard method of calculus.</u> Approximate curve by inscribed broken lines, take arc length to be limit of lengths for these approximations as distances between consecutive points goes to zero.



First approximation

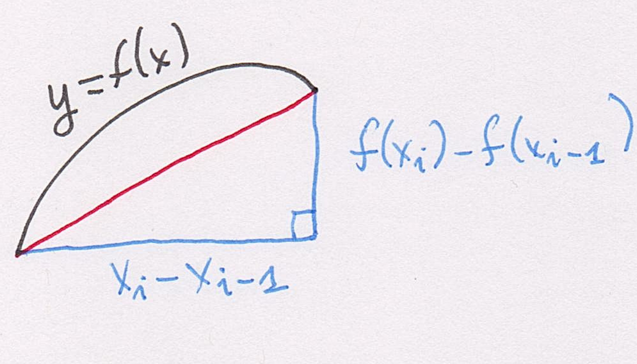
Second (better approximation)

<u>Typical idea</u> Split the interval $[a,b]$ into n equal pieces $a = x_0 < x_1 < \dots < x_n = b$

$$x_i - x_{i-1} = \frac{b-a}{n}.$$

Length of broken line from $(x_0, f(x_0))$ to $(x_1, f(x_1))$, then $(x_1, f(x_1))$ to $(x_2, f(x_2))$, etc. is given by

$$\sum \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}$$



The mean value thm. $\Longrightarrow$ this sum is

$$\sum \sqrt{(x_i - x_{i-1})^2 + f'(c_i)^2 (x_i - x_{i-1})^2} =$$

$$\sum (x_i - x_{i-1}) \sqrt{1 + f'(c_i)^2}.$$

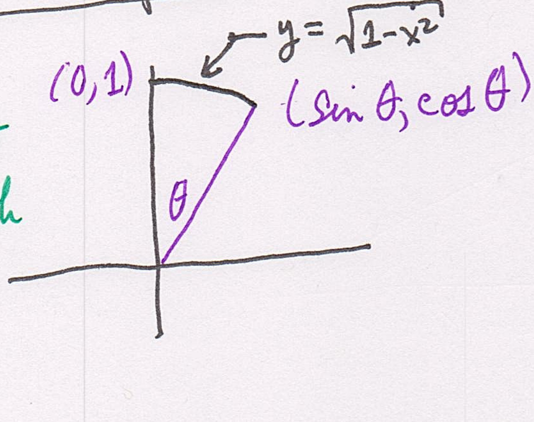
As $n \to \infty$ these expressions go to

① the real arc length

② the definite integral $\int_a^b \sqrt{1+f'(x)^2}\,dx$.

So $\text{Length} = \int_a^b \sqrt{1+(y')^2}\,dx$.

<u>Example</u> $f(x) = \sqrt{1-x^2}$



We expect the arc length to be θ.

So we want this from $x=0$ to $x = \sin\theta$.

If $y = \sqrt{1-x^2}$, then $y' = \frac{-x}{\sqrt{1-x^2}}$, so

$$1 + (y')^2 = 1 + \frac{x^2}{1-x^2} = \frac{1}{1-x^2}$$ and

$$\text{Length} = \int_0^{\sin\theta} \frac{dx}{\sqrt{1-x^2}} = \arcsin(\sin\theta) = \theta,$$

which is what we should have expected!

<u>Another example</u> Let $f(x) = \frac{2}{3}\sqrt{x^3}$,

$1 \leq x \leq a$. $\quad y = \frac{2}{3}x^{3/2} \Rightarrow y' = x^{1/2} \Rightarrow$

$$\sqrt{1+(y')^2} = \sqrt{1+x} \Rightarrow L = \int_1^a \sqrt{1+x}\, dx \underset{u=x+1}{=}$$

$$\int_2^{a+1} \sqrt{u}\, du = \frac{2}{3} u^{3/2} \Big|_2^{a+1} = \frac{2}{3}\left(a^{3/2} - 2^{3/2}\right).$$

Most examples do not integrate out so easily, or at all!

<u>Example</u> $f(x) = x^3 \Longrightarrow$

$$L = \int_a^b \sqrt{1+9x^4}\,dx$$ but there is no

way of writing $\int \sqrt{1+9x^4}\,dx$ in terms of standard functions from first year calculus (but methods from 9 Core helpful).