Second Order Terms in Riemann-Like Sums

Setting ① $M > 0$ constant

② If $\Gamma: a = x_0 < x_1 \ldots < x_N = b$ is a partition of the closed interval $[a, b]$, the norm $|\Gamma|$ of the partition equals $\max_{1 \le i \le N} \{\Delta x_i := x_i - x_{i-1}\}$.

Theorem. Given a partition Γ, let K_i $i = 1, \ldots, N$ be chosen so that $|K_i| \le M$.

Then $\lim_{|\Gamma| \to 0} \sum K_i \Delta x_i^2 = 0$; in other words

for all $\varepsilon > 0$ there is some $\delta > 0$ s.t. for all Γ with $|\Gamma| < \delta$ and all choices of K_i we have

$$\left|\sum K_i \Delta x_i^2\right| < \varepsilon.$$

Proof. Start out with

$$\left|\sum K_i \Delta x_i^2\right| \le \sum |K_i| \cdot \Delta x_i^2 \le$$

$$\sum |K_i| \cdot \Delta x_i \cdot |\Gamma|.$$

The last term is $\leq$

$$\sum M \cdot \Delta x_i \cdot |\Gamma| = M(b-a) \cdot |\Gamma|.$$

Hence $|\Gamma| < \frac{\varepsilon}{M \cdot (b-a)} \Longrightarrow$

$$\left|\sum K_i \Delta x_i^2\right| < \varepsilon,$$ which is what we need to ~~proof~~ prove. ■