Some items from the lectures on Colley, Section I.3

NORMAL COMPONENTS. On pages 21 and 22 of the text there is a discussion of the projection of one vector $\mathbf{b}$ onto a given nonzero vector $\mathbf{a}$. In fact, the vector $\mathbf{b}$ can be written (or in the language of physics "resolved") into a sum $\mathbf{b}_0 + \mathbf{b}_1$, where $\mathbf{b}_0$ is the projection as defined on page 22 and $\mathbf{b}_1$ is perpendicular or **normal** to $\mathbf{a}$.

In order to justify this assertion, we need to show that the vector

$$\mathbf{b}_1 \quad = \quad \mathbf{b} \ - \ \mathbf{b}_0 \quad = \quad \mathbf{b} \ - \ \left(\frac{\mathbf{b}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a}$$

is perpendicular to

$$\mathbf{b}_0 \quad = \quad \left(\frac{\mathbf{b}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a}$$

where as before $\mathbf{a}$ is nonzero. This is actually fairly simple to do, for we need only show that $\mathbf{b}_1 \cdot \mathbf{a} = 0$. To see this, use the first displayed equation to substitute for $\mathbf{b}_1$, so that

$$\mathbf{b}_1 \cdot \mathbf{a} \quad = \quad \left(\mathbf{b} \ - \ \left(\frac{\mathbf{b}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a}\right)\cdot\mathbf{a} \quad = \quad (\mathbf{b}\cdot\mathbf{a}) \ - \ \left(\frac{\mathbf{b}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\right)(\mathbf{a}\cdot\mathbf{a})$$

and if we simplify the right hand side we see that it is equal to zero, which is what we wanted to show.

PARALLEL VECTORS. We often say that two nonzero vectors are *parallel* if each is a nonzero multiple of the other.

Answers to selected exercises from Colley, Section 1.3

8. The angle is the arc cosine of $-1/(3\sqrt{3})$.

12. The perpendicular projection is zero.

14. The answer is the negative of the original vector divided by its length, and this is $\frac{1}{\sqrt{5}} \cdot (1, 0, -2)$.

20. The formula for perpendicular projections shows that $\mathbf{F}_1$ is equal to

$$\left(\frac{\mathbf{F}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a}$$

and since $\mathbf{F}\cdot\mathbf{a} = 2$ while $\mathbf{a}\cdot\mathbf{a} = 17$, we have that $\mathbf{F}_1 = \frac{2}{17} \cdot (4, 1)$. — The vector $\mathbf{F}_2$ is equal to $\mathbf{F} - \mathbf{F}_1$, and it is given by $\frac{9}{17} \cdot (1, -4)$.

24. First of all, if $\mathbf{m}$ is the midpoint of $\mathbf{x}$ and $\mathbf{y}$, then as on page 24 of the text we have $\mathbf{m} - \mathbf{x} = \frac{1}{2}(\mathbf{y} - \mathbf{x})$, and if we solve for $\mathbf{m}$ we see that $\mathbf{m} = \frac{1}{2}(\mathbf{x} + \mathbf{y})$.

Now let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ be four points, no three of which lie on a line. Then we have $\mathbf{m}_1 = \frac{1}{2}(\mathbf{a}+\mathbf{b})$, $\mathbf{m}_2 = \frac{1}{2}(\mathbf{b}+\mathbf{c})$, $\mathbf{m}_3 = \frac{1}{2}(\mathbf{c}+\mathbf{d})$, and $\mathbf{m}_4 = \frac{1}{2}(\mathbf{d}+\mathbf{a})$. We need to show that the vectors $\mathbf{m}_2 - \mathbf{m}_1$ and $\mathbf{m}_4 - \mathbf{m}_3$ are nonzero and parallel, and likewise for the vectors the vectors $\mathbf{m}_4 - \mathbf{m}_1$ and $\mathbf{m}_2 - \mathbf{m}_3$.

If, say $\mathbf{m}_2 - \mathbf{m}_1$ is zero, then $\mathbf{m}_2 = \mathbf{m}_1$ and hence the lines $\mathbf{ab}$ and $\mathbf{bc}$ have two points in common (namely, $\mathbf{b}$ and the common midpoint). Since no three of the original points lie on a single line, this cannot happen, and hence $\mathbf{m}_2 - \mathbf{m}_1$ is nonzero. Similar considerations show that each of the other three differences of midpoints must be nonzero.

Finally, direct computation shows that

$$\begin{aligned} \mathbf{m}_2 - \mathbf{m}_1 &= \tfrac{1}{2}(\mathbf{b} + \mathbf{c}) - \tfrac{1}{2}(\mathbf{a} + \mathbf{b}) = \tfrac{1}{2}(\mathbf{c} - \mathbf{a}) \\ \mathbf{m}_4 - \mathbf{m}_3 &= \tfrac{1}{2}(\mathbf{a} + \mathbf{d}) - \tfrac{1}{2}(\mathbf{c} + \mathbf{d}) = \tfrac{1}{2}(\mathbf{a} - \mathbf{c}) \end{aligned}$$

so that the difference vectors are $\pm$ each other and thus parallel. — Similar considerations imply that the other two vectors are also parallel.