Some items from the lectures on Colley, Section I.4

GEOMETRY OF CROSS PRODUCTS. One can use the algebraic definition of cross product to show that the cross product of two nonzero vectors is perpendicular to both of them (since the cross product of any vector and the zero vector is zero, the statement holds if one factor is zero, but this is less interesting). The key point is that if $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are 3-dimensional vectors, then the triple product $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$ is just the 3×3 determinant whose rows are the given ones (in alphabetical order). The latter is close to, but not quite, the formula on page 33 of the text; the latter states that the determinant is equal to $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$. However, basic properties of determinants show that the determinant does not change if we take the vectors in the order $\mathbf{v}$, $\mathbf{w}$ $\mathbf{u}$, and thus we have

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) \quad = \quad (\mathbf{v}\times\mathbf{w})\cdot\mathbf{u} \quad = \quad (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$$

which yields our version of the determinant formula.

We now need another fundamental property of determinants which can be checked directly: *If two rows are the same, then the determinant is zero.* This and the precedind discussion immediately imply that both $\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b})$ and $\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b})$ are zero, so that $\mathbf{a}$ and $\mathbf{b}$ are both perpendicular to $\mathbf{a}\times\mathbf{b}$. A more detailed study shows that if $\mathbf{a}$ and $\mathbf{b}$ are not scalar multiples of each other then the cross product must be nonzero, but we shall not verify this here.

Example 1. Find the cross product of $(1,-2,1)$ and $(3,1,-2)$.

Solution. Use the 3×3 determinant formula for the cross product:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 3 & 1 & -2 \end{vmatrix}$$

In coordinates this can be rewritten as

$$\left(\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ -2 & 3 \end{vmatrix}, \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} \right)$$

and if we evaluate the 2×2 determinants giving the coordinates we obtain the vector $(3,5,7)$. For multistep computations like this one it is usually a good idea to check the answer, say by verifying that it is actually perpendicular to the original two vectors. Doing this is left to the reader as an exercise.

Example 2. Find a unit vector perpendicular to both $(2,2,1)$ and $(2,-1,-2)$.

Solution. There are two parts. The first is to find some vector perpendicular to both of the given ones, and this can be done by taking the cross product of these vectors. Once this is done, the final step is to divide the result by its length to get a unit vector.

Once again we use the 3×3 determinant formula for the cross product:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 1 \\ 2 & -1 & -2 \end{vmatrix}$$

In coordinates this can be rewritten as

$$\left(\begin{vmatrix} 2 & 1 \\ -1 & -2 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix}, \begin{vmatrix} 2 & 2 \\ 2 & -1 \end{vmatrix} \right)$$

and if we evaluate the 2×2 determinants giving the coordinates we obtain the vector $(-3, 6, -6)$. At this point one should probably pause to check that the result is indeed perpendicular to the original two vectors.

To find a unit vector, we must first compute the length of the vector obtained above. This length is equal to

$$\sqrt{3^2 + 6^2 + 6^2} \quad = \quad \sqrt{9 + 36 + 36} \quad = \quad \sqrt{81} \quad = \quad 9$$

and hence the desired unit vector is equal to

$$\pm \frac{1}{9} \cdot (-3, 6, 6) \quad = \quad \pm \left(-\frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right) \ .$$

Answers to selected exercises from Colley, Section 1.4

6. Evaluating the cross product using the 3×3 determinant formula (3) on page 30 of the text, we see that the cross product is

$$\left(\begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix} \right) \quad = \quad (-3, 2, 5) \ .$$

12. We first find a nonzero vector perpendicular to the given ones by taking their cross product, and the usual formula shows that this cross product $(2, -1, 3) \times (1, 0, 1)$ is equal to

$$\left(\begin{vmatrix} 1 & -3 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} -3 & 2 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} \right) \quad = \quad (1, -5, -1) \ .$$

To get a unit vector perpendicular to the given ones we must divide this cross product by its length, which is $\sqrt{27}$, so that the vector we want is (plus or minus)

$$\frac{1}{\sqrt{27}} \cdot (1, -5, -1) \ .$$

26.

(a) The expression defines a vector.

(b) The expression is not defined because the dot product of two vectors is a scalar and one cannot take the dot product of a scalar and a vector.

(c) The expression is not defined because dot products are scalars and the cross product of two scalars is not defined.

(d) The expression defines a scalar.

(e) The expression not defined because the dot product of the first two terms is a scalar, the cross product of the last two terms is a vector, and the cross product of a scalar and a vector is not defined.

(f) The expression defines a vector.

(g) The expression defines a scalar.

(h) The expression defines a vector.