Some items from the lectures on Colley, Section I.5

Example. Find a linear equation defining the plane through the points $(2,1,1)$, $(0,4,1)$ and $(-2,1,4)$.

Solution. Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be the three vectors given above. Then the plane we want has normal direction which is perpendicular to $\mathbf{b}-\mathbf{a}$ and $\mathbf{c}-\mathbf{a}$, and this normal direction $\mathbf{N}$ must be the cross product of the latter two difference vectors. Thus the vector form for an equation defining the plane is given by

$$\big((\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})\big)\cdot\mathbf{x} \quad = \quad \big((\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})\big)\cdot\mathbf{a}$$

where $\mathbf{x}=(x,y,z)$. Now if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are three 3−dimensional vectors, then $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$ is equal to the 3×3 determinant whose rows are given by these vectors in the order $\mathbf{w}$, $\mathbf{u}$, $\mathbf{v}$; standard properties of 3×3 determinants show that this is equal to the 3×3 determinant whose rows are given by the same vectors taken in alphabetical order.

Substituting the known values for the original three vectors, we have $\mathbf{b}-\mathbf{a}=(-4,0,3)$ and $\mathbf{c}-\mathbf{a}=(-2,3,0)$, so the previous equation defining the plane can be written using determinants as

$$\begin{vmatrix} x & y & z \\ -2 & 3 & 0 \\ -4 & 0 & 3 \end{vmatrix} \quad = \quad \begin{vmatrix} 2 & 1 & 1 \\ -2 & 3 & 0 \\ -4 & 0 & 3 \end{vmatrix}$$

and if we evaluate the determinants on both sides of the displayed equation we obtain the ordinary linear equation

$$9x \;+\; 6y \;+\; 12z \quad = \quad 18 \;+\; 16 \;+\; 12 \quad = \quad 36$$

and if we divide each side by the common factor of 3 we obtain the equation $3x+2y+4z=12$. As usual, it is always good to check that the answer is correct, and in this case one can do so by verifying that all three of the original points satisfy the equation we have obtained.

Answers to selected exercises from Colley, Section 1.5

8. The statement of the problem indicates that the two planes have a point in common, and we can use this without explicitly finding the common point. In any case, the normal direction to the plane is perpendicular to the directions of the two given lines, which are $(1,3,5)$ and $(-1,3,-2)$, so it is given by the cross product, which we can compute as in the preceding section. Its value is $(21,3,-6)=3\cdot(7,1,-2)$. Thus the equation of the plane is $7x+y-2z=7a+b-2c$, where (a,b,c) is any point on the plane. If we take $(2,-5,1)$, which is on the first line (hence also on the plane), the right hand side becomes 7 and hence the equation for the plane is $7x+y-2z=7$.

10. The direction of the line is given by the coefficient vector $(2,-3,5)$, and hence parametric equations for the line are given by

$$\big(\,x(t),\; y(t),\; z(t)\,\big) \quad = \quad (5+2t, -3t, 6+5t)\;.$$

16. Let A, B and C be the three points. Then the directions of the lines AB and AC are given by $(A-B)$ and $(A-C)$, so that parametric equations for the plane are given by $A+u(B-A)+v(C-A)$, where u and v rund through all real numbers. If we substitute for the three vectors using the data in the problem we see that the parametric coordinates are $(-7u + v, u - 2v + 2, -4u + v + 1)$.

18. To find to independent directions in the plane we need to find two vectors which are perpendicular to the plane's normal vector, which is $(2, -3, 5)$. One way to do this is to set one of the coordinates in the reduced equation $2u - 3v + 5w = 0$ equal to zero and solve for one of the remaining coordinates in terms of the other. For example, if we set the first coordinate equal to zero then we get that $v = \frac{5}{3}w$, and any $(0, v, w)$ satisfying this condition will be perpendicular to the plane's normal. Let's take $(0, 5, 3)$. Likewise, if we set the third coordinate equal to zero we obtain the solution $(3, 2, 0)$. To find a point on the plane, set two of the coordinates equal to zero. For example, this shows that $(0, 0, 6)$ lies on the plane. These computations show that parametric equations for the original plane are given by $(3v, 5u + 2v, 3v + 6)$.