Some items from the lectures on Colley, Section I.6

CAPSULE SUMMARY OF MATRICES. Matrices are rectangular arrays into which one can insert numbers or other mathematical objects. An $m \times n$ matrix with real entries is an array of real numbers with m rows and h columns. Usually one writes these numbers with notation like $a_{i,j}$, where i denotes the row number and j denotes the column number, and often one abbreviates this to a_{ij}. So if we are discussing matrices, the spoken form of a_{11} is "a-one-one" and not "a-eleven."

One can add two matrices of the same size by adding the corresponding entries, and similarly one can take the scalar product of a scalar and a matrix by multiplying the respective entries by the given scalar.

Rules for matrix multiplication. The formula for matrix multiplication probably looks unfamiliar to someone who has not seen it before, so we shall give some World Wide Web links which give graphic illustrations.

Suppose that A and B are matrices such that AB is defined. Then the number of columns in A is equal to the number of rows in B. To find the (i, j) entry of the product AB, start by putting your left index finger on $a_{i,1}$ and your right index finger on $b_{1,j}$. Multiply these numbers together and write the answer down as the first entry in a sequence of terms to be added. Next, put your left index finger on $a_{i,2}$ and your right index finger on $b_{2,j}$. Multiply these numbers together and write the answer down as the next entry in the sequence of terms to be added. Continue this way until all terms in the i^{th} row of A and the j^{th} column of B have been exhausted. Now add up all the numbers in the column, and the result is the (i, j) entry of AB.

Here is an example in which A is 3×2 and B is 2×2, so that the product AB is 3×2. Note that one cannot define the product BA in this case because the number of columns in B is 2 and the number of rows in A is 3.

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}, \qquad B = \begin{pmatrix} 7 & 8 \\ 9 & 10 \end{pmatrix}$$

$$AB = \begin{pmatrix} (1\cdot 7)+(4\cdot 9) & (1\cdot 8)+(4\cdot 10) \\ (2\cdot 7)+(5\cdot 9) & (2\cdot 8)+(5\cdot 10) \\ (3\cdot 7)+(6\cdot 9) & (3\cdot 8)+(6\cdot 10) \end{pmatrix} = \begin{pmatrix} 43 & 48 \\ 59 & 66 \\ 75 & 84 \end{pmatrix}$$

An animated example of this procedure — with the right index finger replaced by the left thumb — is given on the following site:

`http://www.purplemath.com/modules/mtxmult.htm`

The file `weblinks16.pdf` in the course directory contains clickable versions for this and all the other web links cited here.

Here are two similar links without animation:

`http://en.wikipedia.org/wiki/Matrix_multiplication`

`http://www-unix.mcs.anl.gov/dbpp/text/node45.html`

Here are some more advanced links which also contain self-tests:

`http://www.algebralab.org/lessons/lesson.aspx?file=Algebra_matrices_multiplying.xml`

`http://www.intmath.com/Matrices-determinants/4_Multiplying-matrices.php`

Finally, here is a link to an animated YouTube video (with sound):

`http://www.youtube.com/watch?v=sYlOjyPyX3g`

ORDER OF MATRIX MULTIPLICATION. On page 53 it is noted that the product of two matrices A and B depepends strongly on the order in which the factors are written, even in cases where both matrices are $n \times n$ so that AB and BA are both defined and have the same dimensions. The most convincing way to show this is to give specific examples of 2×2 matrices for which the two products differ, and here they are:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

COURSE COVERAGE OF DETERMINANTS. In previous sections of the text we assumed familiarity with the standard rules for computing the determinants of 2×2 and 3×3 matrices that are covered in "college [or second year high school] algebra" or precalculus courses. Here are an animated site and a YouTube site reviewing the rules in the 3×3 case:

`http://www.college-cram.com/study/algebra/presentations/226`

`http://www.youtube.com/watch?v=kau7pUqmXgU`

In this course we shall **NOT** be covering the rules for computing $n \times n$ determinants for $n > 3$; however, they may (and quite likely will) be needed in Mathematics 10B. For possible future reference, it is worth noting that the 2×2 determinant formula $ad - bc$ (or $\searrow - \nearrow$) and the frequently used 3×3 formula

$$\searrow + \searrow + \searrow - \nearrow - \nearrow - \nearrow$$

do **NOT** have counterparts for higher order determinants. the formula for an $n \times n$ determant has $1 \cdot 2 \ \cdot ... \cdot n = n!$ terms and is describable using the expansion by minors in the text; in particular, the formula for a 4×4 determinant involves $4! = 24$ terms and not just the 8 terms that one would obtain by taking the formula

$$\searrow + \searrow + \searrow + \searrow - \nearrow - \nearrow - \nearrow - \nearrow$$

which one might expect by extrapolating from the previously described 3×3 formula.

Answers to selected exercises from Colley, Section 1.6

8. The answer is

$$\left(\frac{11}{32}, -\frac{11}{32}, \frac{77}{32}, \frac{11}{16} \right) .$$

12. We assume that $\mathbf{a}$ and $\mathbf{b}$ are both nonzero.

The inequality $|\mathbf{a} - \mathbf{b}| > |\mathbf{a} + \mathbf{b}|$ is equivalent to $|\mathbf{a} - \mathbf{b}|^2 > |\mathbf{a} + \mathbf{b}|^2$, and expanding the latter two expressions implies that the latter is true if and only if $-2(\mathbf{a} \cdot \mathbf{b}) > 2(\mathbf{a} \cdot \mathbf{b})$. Since a real number x satisfies $-2x > 2x$ if and only if x is negative, this implies that $\mathbf{a} \cdot \mathbf{b}$ is negative, and by the formula relating dot products and angle cosines, this is true if and only if the measurement of the angle $\angle \mathbf{a0b}$ is between $90°$ and $180°$.

18. We have

$$DB = \begin{pmatrix} -4 & 9 & 5 \\ -8 & 9 & 10 \end{pmatrix} .$$

28.(a) If A is the 2×2 matrix with a 1 in the $(1,1)$ position and zeros elsewhere, B is the 2×2 matrix with a 1 in the $(2,2)$ position and zeros elsewhere, then $A + B$ is the so-called 2×2 *identity matrix*

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and $\det(A+B) = 1$, but $\det A = \det B = 0$ so that $\det(A+B) = \det A + \det B = 0 + 0 = 0$.