Some items from the lectures on Colley, Section I.7

INTERPRETING PLANE POLAR COORDINATES. Ordinary rectangular coordinates can be used to describe the position of one point P with respect to a fixed reference point Q; for example, if P has rectangular coordinates $(2, 3)$ and Q has rectangular coordinates $(0, 0)$ then P is 2 units east and 3 units north of Q. Of course, if P has coordinates (x, y) where x or y is negative, then $x < 0$ means $|x|$ units west, and similarly $y < 0$ means $|y|$ units south. The idea of polar coordinates is to give two numbers r and θ so that the distance between the points is r units and the direction of the displacement from Q to P makes and angle lf θ (usually in radians) with respect to some fixed reference direction. In ordinary geography, the reference direction is usually north, but in mathematical polar coordinates the reference direction is straight east and the angle is measured in a *counterclockwise* sense from straight east.

INTERPRETING CYLINDRICAL COORDINATES. These are a mixture of plane polar coordinates and rectangular coordinates, where r and θ are related to x and y exactly as for polar coordinates, and z simply goes to itself.

INTERPRETING SPHERICAL COORDINATES. The coordinate ρ measures the distance from the point (x, y, z) to the origin. The remaining θ and ϕ coordinates are like longitude and latitude coordinates on the surface of the sphere. Visualizing — or looking at and touching — an ordinary globe depicting the surface of the earth is extremely useful when working problems involving spherical coordinates. Here are some web sites that might provide useful background; clickable links for all web references in this document are in the file `weblinks17.pdf` in the course directory.

`http://nationalatlas.gov/articles/mapping/a_latlong.html`

`http://en.wikipedia.org/wiki/Geographic_coordinate_system`

Although it is often helpful to think about spherical coordinates in terms of latitude and longitude, it is also important to recognize some major differences between the spherical coordinates θ and ϕ and ordinary longitude and latitude. The longitudinal coordinate θ is scaled so that the $0°$ meridian points straight east and the longitudes increase in the counterclockwise direction, and as with the polar coordinate θ we know that θ and $\theta + 2k\pi$ represent the same meridian for every integer k. The latitudinal coordinate ϕ is scaled so that the North Pole $(0, 0, 1)$ corresponds to $\phi = 0$, the value of ϕ increases as one moves southward, and the South Pole $(0, 0, -1)$ corresponds to $\phi = \pi$; for every integer k the values ϕ and $\phi + 2k\pi$ determine the same latitude, and the values *phi* and $2\pi - \phi$ also determine the same latitude.

The examples in the following online link can be used as practice for converting ordinary geographical coordinates into spherical coordinates:

`http://academic.brooklyn.cuny.edu/geology/leveson/core/linksa/longlatquiz1.html`

The conversions will worked out in the answers at the end of this document.

EXAMPLES.

1. Convert the polar coordinates $(3, \frac{5\pi}{4})$ to rectangular coordinates.
Solution. We know that $r = 3$ and $\cos \frac{5\pi}{4} = \cos 225° = -\sqrt{2}/2 = \sin \frac{5\pi}{4}$, so that $x = 3\cos 135° = -3\sqrt{2}/2$ and also $x = 3\sin 135° = -3\sqrt{2}/2$.

2. Convert the polar coordinates $(-2, \frac{11\pi}{6})$ to rectangular coordinates.

Solution. We use the same conversion formula regardless of whether or not r is positive. Now $\cos\frac{11\pi}{6} = \cos -30^\circ = \sqrt{3}/2$ and $\sin\frac{11\pi}{6} = \sin -30^\circ = -1/2$. Therefore $x = -2\sqrt{3}/2 = -\sqrt{3}$ and $y = -2\cdot(-\frac{1}{2}) = 1$.

3. Convert the rectangular coordinates $(3, -1)$ to polar coordinates. You need only give one set of polar coordinates.
Solution. Since r is given explicitly as $\sqrt{x^2+y^2}$, it is easiest to begin by finding r. In this example we find that $r = \sqrt{10}$ (fill in the details!). We then have $\cos\theta = x/r = 3/\sqrt{10}$ and $\sin\theta = y/r = -1/\sqrt{10}$ and hence θ must lie in the fourth quadrant. This means that

$$\theta \quad = \quad \text{Arc sin}\,\frac{-1}{\sqrt{10}}$$

so that one set of polar coordinates is given by

$$\left(\sqrt{10},\ \text{Arc sin}\,\frac{-1}{\sqrt{10}}\right) .$$

The ordered pair

$$\left(\sqrt{10},\ \text{Arc tan}\,\frac{-1}{3}\right)$$

also gives a correct answer to the problem.

4. Convert the rectangular equation $xy = 4$ to polar coordinates.
Solution. Use the polar coordinate formulas for x and y in terms of r and θ to see that $4 = xy = (r\cos\theta)\cdot(r\sin\theta)$, which can also be rewritten in the forms

$$\frac{r^2\sin 2\theta}{2} \quad = \quad 4 \qquad \text{or} \qquad r^2\sin 2\theta \quad = \quad 8 .$$

5. Convert the rectangular equation $y^2 = 9x$ to polar coordinates.
Solution. Again use the polar coordinate formulas for x and y. This yields the equation $r^2\sin^2\theta = 9r\cos\theta$. One might try to simplify this by cancelling r from each side and obtaining $r\sin^2\theta = 9\cos\theta$, **BUT** when doing so it is important to see whether the solutions $r = 0$ is lost during this process (cancellation requires that r must be nonzero). Fortunately, in this case it is not lost, for the ordered pair $(0, \frac{1}{2}\pi)$ satisfies the equation $r\sin^2\theta = 9\cos\theta$.

6. Convert the polar equation $r = \sin\theta$ to rectangular coordinates.
Solution. The idea of such problems is usually to multiply both sides of the equation so that one obtains terms like r^2, $r\sin\theta$ and $r\cos\theta$ which are easily expressible in rectangular form. In this case, if we multiply both sides by r we obtain $r^2 = r\sin\theta$, so that $x^2 + y^2 = y$, and if we move the y term to the right side and complete the square we obtain

$$x^2 \ + \ \left(y - \frac{1}{2}\right)^2 \quad = \quad \frac{1}{4}$$

which is the equation of a circle with center $(0, \frac{1}{2})$ and radius $\frac{1}{2}$.

7. Convert the polar equation $r = 3\sec\theta$ to rectangular coordinates.
Solution. Use the same idea. First of all, if we multiply both sides by $\cos\theta$ we obtain $r\cos\theta = 3$, so that the equation reduces to that of the horizontal line $x = 3$.

8. Convert the cylindrical coordinate equation $r = \frac{1}{2}\,z$ to rectangular coordinates.
Solution. Square both sides to get an equation involving r^2, so that we have $r^2 = \frac{1}{4}\,z^2$, which can be rewritten in the form $4(x^2 + y^2) = z^2$, the equation of a cone.

9. Convert the rectangular coordinates $(1, 1, 1)$ to spherical coordinates. You need only give one set of spherical coordinates.
Solution. Since ρ is given by $\sqrt{x^2 + y^2 + z^2}$, one can begin by computing directly that $\rho = \sqrt{3}$. Usually in these problems the next step is to compute ϕ using the equation $z = \rho\cos\phi$; in this example substitution yields $1 = \sqrt{3}\cos\phi$, so that

$$\cos\phi \quad = \quad \frac{1}{\sqrt{3}} \qquad \text{or} \qquad \phi \quad = \quad \text{Arc cos}\,\frac{1}{\sqrt{3}}\ .$$

We can now combine our results for ρ and ϕ with the equations $x = \phi\cos\theta\sin\phi$ and $y = \phi\sin\theta\sin\phi$ to find θ. First of all, we have

$$\sin\phi \quad = \quad \sqrt{1 - \cos^2\phi} \quad = \quad \sqrt{1 - \tfrac{1}{3}} \quad = \quad \sqrt{\tfrac{2}{3}}$$

which yields the equation(s)

$$1 \quad = \quad \sqrt{3}\,\sin\theta\sqrt{\frac{2}{3}} \quad = \quad \sqrt{2}\,\sin\theta$$

so that $\sin\theta = \frac{1}{2}\sqrt{2}$. We also have a similar equation with $\cos\theta$ replacing $\sin\theta$, and from this we see that $\cos\theta = \frac{1}{2}\sqrt{2}$. If we combine these, we see that θ must be $\frac{1}{4}\pi$ (up to adding an integer multiple of 2π, as usual).

10. Convert the spherical coordinates $(4, \frac{1}{6}\pi, \frac{1}{4}\pi)$ to rectangular coordinates.
Solution. In this problem it is only necessary to substitute the given values for ρ, θ and ϕ into the formulas expressing x, y and z in terms of spherical coordinates. Recalling that $\frac{1}{6}\pi = 30^\circ$ and $\frac{1}{4}\pi = 45^\circ$, we have

$$\begin{aligned} x \quad &= \quad 4\,\cos 30^\circ\ \sin 45^\circ \quad = \quad 4 \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} \quad = \quad \sqrt{6} \\ y \quad &= \quad 4\,\sin 30^\circ\ \sin 45^\circ \quad = \quad 4 \cdot \frac{1}{2} \cdot \frac{\sqrt{2}}{2} \quad = \quad \sqrt{2} \\ z \quad &= \quad 4\,\cos 45^\circ \quad = \quad 4 \cdot \frac{\sqrt{2}}{2} \quad = \quad 2 \cdot \sqrt{2} \end{aligned}$$

as the answer. One way to check the accuracy of this result is to verify that $\rho = \sqrt{6 + 2 + (4 \cdot 2)} = \sqrt{16} = 4$.

11. Convert the spherical coordinates $(9, \frac{1}{4}\pi, \pi)$ to rectangular coordinates.
Solution. One can follow the same procedure as before. In this case $x = y = 0$ because $\sin\pi = 0$, and $z = 9\cos\pi = -9$. Note that we get the same rectangular coordinates for all choices of θ in this case.

12. Convert the equation $x^2 + y^2 + z^2 = 2z$ to cylindrical coordinates, and describe the set of points which satisfy this equation.
Solution. This can be done fairly easily because $r^2 = x^2 + y^2$, so that the equation reduces to $r^2 + z^2 = 2z$, or equivalently $r^2 + z^2 - 2z = 0$. Completing the square, we can rewrite this as

$$r^2 + (z - 1)^2 = 1$$

so that the equation defines a sphere whose radius is 1 and whose center is $(0, 0, 1)$.

13. Describe the region defined by the points whose cylindrical coordinates satisfy $0 \le \theta \le \frac{\pi}{2}$, $0 \le r \le 2$ and $0 \le z \le 4$.
Solution. This is a piece of a cylinder obtained by cutting the cylinder into four equal pieces which look like slices of a cheesecake.

14. Describe the region defined by the points whose cylindrical coordinates satisfy $0 \le \theta \le 2\pi$, $0 \le r \le a$ and $r \le z \le a$, where $a > 0$.
Solution. The third chain of inequalities yields $r^2 \le z^2$ or equivalently $x^2 + y^2 \le z^2$, and the first chain shows that θ can essentially be anything. This means that the figure defined by the inequalities is given by rotating a figure in the xz-plane about the z-axis. We can use this to reduce the analysis of the original figure to the analysis of the portion of the figure which lies in the xz-plane. The latter has equation $y = 0$, so the relevant piece of the figure is defined by $0 \le |x| \le a$ and $|x| \le z \le a$. This is a solid isosceles right triangular region whose vertices are the origin and the points $(\pm a, 0, a)$, and the figure we are trying to describe must be a cone formed by revolving this solid triangular region about the z-axis.

15. Describe the region defined by the points whose spherical coordinates satisfy $0 \le \theta \le 2\pi$, $\frac{\pi}{4} \le \phi \le \frac{\pi}{2}$ and $0 \le \rho \le 1$.
Solution. Once again this is obtained by rotating a figure in the xz-plane about the z-axis. The second equation indicates that the latitude is between the equator and 45° North. So the figure in the xz-plane consists of all points $(x, 0, z)$ such that $|x| \le z$ and $x^2 + z^2 \le 1$, which is given by a pair of solid pie slices in the plane whose radii are equal to 1 and whose vertices are $(0, 0, 0)$ and $(\pm \frac{1}{2}\sqrt{2}, 0, \frac{1}{2}\sqrt{2})$.

16. Describe the region defined by the points whose spherical coordinates satisfy $0 \le \theta \le 2\pi$, $0 \le \phi \le \frac{\pi}{6}$ and $0 \le \rho \le a \sec\phi$.
Solution. Once again this is obtained by rotating a figure in the xz-plane about the z-axis. If we multiply the third equation by $\cos\phi$ and use the formula $z = \rho\cos\phi$, we obtain the equation $0 \le z \le a$, and similarly if we apply the cosine function to the second chain of inequalities and muliply everything by ρ we see that

$$z \;=\; \rho\cos\phi \;\ge\; \rho\cos\frac{\pi}{6} \;=\; \rho\,\frac{\sqrt{3}}{2}$$

and since $\rho^2 = x^2 + z^2$ on the xz-plane it follows that

$$x^2 \;=\; \rho^2 - z^2 \;\le\; \rho^2 - \tfrac{3}{4}\rho^2 \;=\; \tfrac{1}{4}\rho^2$$

and hence

$$|x| \;\le\; \frac{1}{2}\rho \;\le\; \frac{z}{\sqrt{3}}$$

so that $z \ge |x| \cdot \sqrt{3}$. Thus the region which is rotated around the z-axis is the solid triangular region defined by the triangle whose sides are the origin and $\pm\left(a/\sqrt{3}, 0, a\right)$, and the solid of revolution in the problem is the associated solid cone.

17. Write the rectangular coordinate equation $4z = x^2 + y^2$ in cylindrical coordinates.
Solution. Since $x^2 + y^2 = r^2$ this reduces to $4z = r^2$, and we can rewrite this further as $r = 2\sqrt{z}$, where $z \ge 0$.

18. Write the cylindrical coordinate equation $\theta = c$ in rectangular coordinates.

Solution. If $\theta \neq \pm\frac{\pi}{2}$ then $y = x \tan\theta$ and thus the equation reduces to $y = L \cdot x$, where $L = \tan c$. On the other hand, if θ is not an integer multiple of π then we have $x = y \cdot \cot\theta$ so that the equation reduces to $x = My$, where $M = \cot c$. Geometrically, the figure defined by the equation is a plane which is perpendicular to the xy-plane and passes through the origin.

Answers to selected exercises from Colley, Section 1.7

2. The rectangular coordinates are $(-3/2, \sqrt{3}/2)$.

4. The coordinates satisfy $r = 4$ and $\tan\theta = \frac{1}{\sqrt{3}}$. Since the point lies in the first quadrant, this means that $\theta = \frac{1}{6}\pi$ and the polar coordinates are $(4, \frac{1}{6}\pi)$.

8. The rectangular coordinates are $(0, \pi, 1)$.

10. The answer is $(2, 2\sqrt{3}, 0)$.

12. The answer is $(-1/\sqrt{8}, \sqrt{3/8}, -1/\sqrt{2})$.

18. The spherical coordinates are $(2, \frac{1}{3}\pi, \frac{1}{2}\pi)$.

20.(a) The equation of the curve implies that $r^2 = 2ar\sin\theta$, which in turn translates into $x^2 + y^2 = 2ay$. The latter can be rewritten as $x^2 + (y^2 - 2ay) = 0$, which is equivalent to $x^2 + (y-a)^2 = a^2$. This is the equation of a circle with center $(0, a/2)$ and radius a.

24. In cylindrical coordinates the equation becomes $z^2 = 2r^2$, which is the equation of a cone. In spherical coordinates the equation becomes $\cos\phi = \pm\sqrt{2/3}$.

26. The figure is a 90° wedge from a solid cylinder, essentially one fourth of a cheesecake or a cheese wheel.

Answers to latitude – longitude quiz

Finally, here are some comments regarding the quiz which appears in one of the Internet sites mentioned earlier.

`http://academic.brooklyn.cuny.edu/geology/leveson/core/linksa/longlatquiz1.html`

In the illustration, the **red dot** is located at 30° West and 60° North, the **orange dot** is located at 15° West and 10° South, the **blue dot** is located at 60° West and 50° North, the **green dot** is located at 45° East and 20° North, and the **pink dot** is located at 30° East and 10° Sorth. If we view the sphere as the set of all points in coordinate 3-space satisfyiing $\rho = 1$, then the spherical coordinates $(\rho\theta, \phi)$ of the dots are given as follows:

The red dot has spherical coordinates $(1, -\frac{1}{6}\pi, \frac{1}{6}\pi)$.
The orange dot has spherical coordinates $(1, -\frac{1}{12}\pi, \frac{5}{9}\pi)$.
The blue dot has spherical coordinates $(1, -\frac{1}{3}\pi, \frac{2}{9}\pi)$,
The green dot has spherical coordinates $(1, \frac{1}{4}\pi, \frac{7}{18}\pi)$.
The pink dot has spherical coordinates $(1, \frac{1}{6}\pi, \frac{5}{9}\pi)$.