Some items from the lectures on Colley, Section 2.1

THE FUNCTION CONCEPT. In mathematics, a function describes a relationship between two types of variable quantities: The *dependent variables* are given in terms of the *independent variables*, and for each choice of independent variables there is **exactly one** value for the dependent variable(s). In single variable calculus there is only one independent variable and one dependent variable, but in multivariable calculus there are ordered lists of 2, 3, or any finite number of dependent or independent variables.

Examples. Polynomial functions of two variables such as $f(x,y) = 2x + 3y$ or $x^2 + xy + y^2$ or $x^3 + 3x^2y + 3xy^2 + y^3$.

The text uses notation like $f : X \to Y$ to denote a function. This means that the independent variable choice comes from the collection of objectx X and the value of the dependent variable belongs to the collection of objects Y; the value is usually called $f(x)$. At the bottom of page 79 there is a discussion and drawing which provide one way of thinking about a function geometrically. Three other important ways of describing a function are by a formula like, say, $\sin^3 x/(y^2 + 1)$, by a table of values (see page 82 of the text), or by means of a graph (see page 83).

If f is a real valued functiton of two real variables, it is often useful to think of the graph of f as a surface in coordinate 3-space consisting of all (x, y, z) such that $z = f(x, y)$. For example, if $f(x, y)$ is the linear function $2x + 3y + 4$, then the graph is given by $z = 2x + 3y + 4$, which is the equatiion of a plane. A more complicated example appears on page 84 of the text. Often it is useful to describe a such a function by looking at level curves of the form $f(x, y) =$ constant, where the constant ranges over all possible values for f. In particular, if f is the previously described linear function, these level curves are the mutually parallel lines in the xy-plane with equations $2x + 3y + 4 = C$, where C can be any real number. Illustrations of more complicated examples appear on pages 85–87.

Caution. Not every surface in coordinate 3-space is the graph of a function. For example, the equation $z^2 = 1$ defines a surface consisting of the two parallel planes $z = \pm 1$, but it is not the graph of a function because for each choice of (x, y) there are two points in the surface of the form (x, y, z) (namely, take $z = \pm 1$), and for the graph of a surface there can be only on (x, y, z) in the graph with given first and second coordinates. The spherical surface defined by the equation $x^2 + y^2 + z^2 = 1$ is a similar but less trivial example; once again, if (x, y, z) satisfies the defining equation, then so does $(x, y, -z)$.

QUADRIC SURFACES. This important class of examples is described at the end of Section 2.1 in the text. The defining equation is given explicitly on page 89. Usually we want to assume some nontriviality condition on the coefficients of the second degree terms; for example, if all the coefficients are zero then the equation is merely a polynomial of lower degree (and hence defines a plane, provided of course that the equation is nontrivial).

It turns out that, up to a suitable notion of geometric congruence, there are only finitely many types of quadrics. Furthermore, under reasonable linear changes of variables such as $u = ax + b$, $v = cy + d$, $w = ez + f$ – where a, b and c are all nonzero – the type of the quadric does not change. A complete list appears in the file `quadrics.pdf` in the course directory, and some of the most important types are described on pages 89–91 of the text. In addition to these, one has the *cylindrical surfaces* for which the defining equation only involves two of the variables x, y, z. For example, if z does not appear in the equation, then the quadric is the cylindrical surface generated by all points on lines which are perpendicular to the xy-plane and meet the latter at the associated curve in the xy-plane (remember that the defining equation only involves x and y, and as such it also defines a plane curve).

Examples from Colley, Section 2.1

41. Use the change of variables $u = x/5$, $v = 4/4$, so that the defining equation becomes $u^2 + v^2 - z^2 = -1$. This can be written in the form $G(u^2 + v^2, z) = -1$, and it follows that the surface is a surface of revolution about the z-axis. The curve which is rotated about this axis is the curve where the uz-plane (with equation $v = 0$) meets the z-axis, and its defining equation is $u^2 - z^2 = -1$. This is a hyperbola which "opens up" around the z-axis and whose two branches lie on opposite sides of the u-axis, and the resulting surface of revolution is a two-sheeted hyperboloid.

46. If we complete squares for the x and y variables we obtain the equivalent defining equation

$$(4x^2+8x+4) - 4 + (y^2-4y+4) - 4 - 4z^2 + 4 = (2x+2)^2 + (y-2)^2 - (2z)^2 - 4 = 0$$

and if we make the substitutions $u = 2x + 2$, $v = y - 2$ and $w = 2z$ the equation becomes

$$u^2 + v^2 - w^2 = 4$$

so that the surface is obtained by rotating the curve $u^2 - w^2 = 4$ in the uw-plane (again with equation $v = 0$) about the w-axis. The latter curve is a hyperbola which "opens up" around the u-axis and whose two branches lie on opposite sides of the w-axis, and the resulting surface of revolution is a one-sheeted hyperboloid.

Answers to selected exercises from Colley, Section 2.1

2. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be defined by $f(x, y) = 2x^2 + 3y^2 - 7$.
(a) The domain of g is $\{(x, y) \in \mathbf{R}^2\}$, and the range of g is $\{z \in \mathbf{R} \mid z \geq -7\}$.
(b) Let the domain of g is $\{(x, x) \in \mathbf{R}^2 \mid x \geq 0\}$.
(c) Take the codomain of g to be the range of g.

4. The domain of f is $\{(x, y) \in \mathbf{R}^2 \mid x + y > 0\}$, and the range of f is $\mathbf{R}$.

6. The domain of g is $\{x \in \mathbf{R}^3 \mid |x| < 2\}$, and the range of g is $\{y \in \mathbf{R} \mid y \geq \frac{1}{2}\}$.

12. For $c > -9$ the level sets are circles centered at the origin of radius $\sqrt{c}$. For $c = -9$ the level set is just the origin. There are no values corresponding to $c < -9$. Note that the curves get closer together, indicating that we are climbing faster as we head out radially from the origin.

18. The surface is a plane. Level sets for which $f(x, y) = c$ are lines $c = 3 - 2x - yory = -2x + (3 - c)$.

28. The level surfaces are planes $x - 2y + 3z = c$.

32. The level surfaces are of the form $y(x - z) = c$. If $c = 0$ we get the union of the xz-plane and the plane $x = z$. If $c \neq 0$ we get the hyperbola in the xy-plane $y = c/x$; this generates the solution surfaces when translated by $m(1, 0, -1)$.

34.(a) F is, of course, not uniquely determined. But if we let $F(x, y, z) = x^2 + xy - xz - 2$, then the surface is the level set $F(x, y, z) = 0$.

(b) $x^2 + xy - xz = 2 is$ equivalent to

$$z = \frac{x2 + xy - 2}{x} = f(x, y) \ .$$

36. The figure is a hyperbolic paraboloid or *saddle surface*. To see this, do a change of variables $u = x/2$, so that the defining equation becomes $z = u^2 - y^2$