Some items from the lectures on Colley, Section 2.1

THE LIMIT CONCEPT. Intuitive notions of limits were already recognized by several Greek mathematicians, most notably Archimedes. During the 17^{th} century the importance of limit concepts in the development of differential and integral calculus became increasingly apparent. However, a logically precise formulation of limits turned out to be more elusive than expected, and it was not until the middle of the 19^{th} century that the standard formal definitions of limits were created. Most users of calculus do not need to know the formal definition of a limit; the latter is absolutely indispensable for ensuring that mathematics rests on a logically sound foundation, but in practice it is usually enough to have an intuitive understanding of limits and to know some basic rules for manipulating expressions involving limits.

The file `limitexamples.pdf` in the course directory contains review material on limits from elementary geometry and single variable calculus, with several animated sites, including YouTube tutorials on limits.

The basic setting for limits in several variables is the following. We suppose that we are given a real valued function f which is defined on some open disk $N_r(\mathbf{a})$ of radius $r > 0$ about some point $\mathbf{a}$ in $\mathbf{R}^n$, except possibly at $\mathbf{a}$ itself. Saying that

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) \;=\; b$$

essentially means that

$$f(\mathbf{x}) \textit{ is close to } b \textit{ if } \mathbf{x} \neq \mathbf{a} \textit{ is sufficiently close to } \mathbf{a}$$

and more precisely, that if $h > 0$ then the value of the function is within h of b provided $\mathbf{x} \neq \mathbf{a}$ is sufficiently close to $\mathbf{a}$. In analogy with single variable limits, these multivariable limits have the following simple properties which allow one to compute many limit values without appealing directly to the definitions; in each of these identities we assume that the limits on the right hand side exist, and the equations mean that the limits on the left hand side also exist.

$$\lim_{\mathbf{x}\to\mathbf{a}} x_i \;=\; a_i \qquad (u_i = i^{\text{th}} \text{ coordinate of } \mathbf{u})$$

$$\lim_{\mathbf{x}\to\mathbf{a}} f(x) \pm g(x) \;=\; \lim_{\mathbf{x}\to\mathbf{a}} f(x) \;\pm\; \lim_{\mathbf{x}\to\mathbf{a}} g(x)$$

$$\lim_{\mathbf{x}\to\mathbf{a}} f(x) \cdot g(x) \;=\; \lim_{\mathbf{x}\to\mathbf{a}} f(x) \;\cdot\; \lim_{\mathbf{x}\to\mathbf{a}} g(x)$$

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{f(x)}{g(x)} \;=\; \lim_{\mathbf{x}\to\mathbf{a}} f(x) \,/\, \lim_{\mathbf{x}\to\mathbf{a}} g(x) \qquad \text{if} \quad \lim_{\mathbf{x}\to\mathbf{a}} g(x) \;\neq\; 0$$

The preceding statements mean that taking limits is compatible with the four basic arithmetic operations, and that the limits of the fundamental coordinate functions are what we would expect. Here are two other important properties:

(**Composite Property**) If we have $\lim_{u\to b} f(u) = c$ and $\lim_{\mathbf{x}\to\mathbf{a}} g(x) = b$, then $\lim_{\mathbf{x}\to\mathbf{a}} f(\,g(x)\,) = c$.

(**Squeeze Property**) If $h_1(\mathbf{x}) \leq f(\mathbf{x}) \leq h_2(\mathbf{x})$ for all $\mathbf{x} \neq \mathbf{a}$ sufficiently close to $\mathbf{a}$ and $\lim_{\mathbf{x}\to\mathbf{a}} h_1(x) = \lim_{\mathbf{x}\to\mathbf{a}} h_2(x) = L$, then $\lim_{\mathbf{x}\to\mathbf{a}} f(x) = L$.

One can use these rules to evaluate nearly every type of limit that arises in an introductory multivariable calculus course.

OPEN AND CLOSED SUBSETS OF EUCLIDEAN n-SPACE. For most purposes, it is useful to think of **open sets** as defined as collections of points satisfying a finite list of strict inequalities $g_i(x_1, x_2, \cdots) > 0$ where each g_i is continuous, and to think of **closed sets** as defined as collections of points satisfying a finite list of non-strict inequalities $f_i(x_1, x_2, \cdots) \geq 0$ where each g_i is continuous. Note that one can view an inequality $f < 0$ equivalently as $-f > 0$, and one can view an equation $f = 0$ as the pair of non-strict inequalities $f \geq 0$ and $-f \geq 0$.

Here are properties of open and closed sets worth noting: If U is an open subset of some Euclidean space and $\mathbf{x} \in U$, then there is some $r > 0$ such that the set of all points $\mathbf{y}$ satisfying $|\mathbf{y} - \mathbf{x}| < r$ is totally contained in U. If F is a closed subset of some Euclidean space ahd we are given a sequence of points $\{ \mathbf{x}_n \}$ in F which converges to a limit vector $\mathbf{y}$ (in other words, for each i the i^{th} coordinates of the $\mathbf{x}_n$ converge to the i^{th} coordinate of $\mathbf{y}$), then $\mathbf{y}$ also lies in F; in other words, f is "closed under taking limits of sequences."

As a rule, subsets defined by combinations of strict and non-strict inequalities are neither open nor closed.

Answers to selected exercises from Colley, Section 2.2

2. This is an annulus which includes all of its boundary points and therefore it is closed.

4. This is a hollowed out sphere which includes its boundary points and therefore it is closed.

6. This is the open infinite cylinder in $\mathbf{R}^3$ and therefore it is open.

8. We can see that

$$\lim_{(x,y)\to(0,0)} \frac{|y|}{\sqrt{x^2+y^2}}$$

does not exist by looking at the limit along the paths $x = 0$ and $y = 0$. In the first case we have

$$\lim_{(0,y)\to(0,0)} \frac{|y|}{\sqrt{x^2+y^2}} \quad = \quad \lim_{(0,y)\to(0,0)} \frac{|y|}{|y|} \quad = \quad 1$$

while in the second we have

$$\lim_{(x,0)\to(0,0)} \frac{|x|}{\sqrt{x^2+y^2}} \quad = \quad \lim_{(0,y)\to(0,0)} \frac{0}{|x|} \quad = \quad 0\ .$$

14. If there was a limit value L, then we would have

$$L \quad = \quad \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \quad = \quad \lim_{(x,0)\to(0,0)} \frac{0}{x^2} \quad = \quad 0$$

and we would also have

$$L \quad = \quad \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \quad = \quad \lim_{(x,x)\to(0,0)} \frac{x^2}{2x^2} \quad = \quad \frac{1}{2}$$

Since we cannot have $L = 0$ and $L = \frac{1}{2}$, the given function does not have a limit at $(0,0)$.

18. One obtains different values depending on the path one takes to $(x, y) = (2, 0)$. For the path $(2, y) \to (2, 0)$ the limit is -1, and for the path $(x, 0) \to (2, 0)$ the limit is 1. Therefore there is no possible value for the limit.

22.(a) We know from the direct geometric argument in single-variable calculus that

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 .$$

(b) If we let $\theta = x + y$ then $\lim_{(x,y)\to(0,0)} x + y = 0$, and therefore by the composite rule we have

$$\lim_{(x,y)\to(0,0)} \frac{\sin x + y}{x + y} = \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 .$$

(c) If we let $\theta = xy$ then $\lim_{(x,y)\to(0,0)} xy = 0$, and therefore by the composite rule we have

$$\lim_{(x,y)\to(0,0)} \frac{\sin xy}{xy} = \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 .$$

28. If we convert to polar coordinates we obtain

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2 + y^2} = \lim_{r \to 0} \frac{r^2 \cos^2 \theta \cdot r \sin \theta}{r^2} = \lim_{r \to 0} r \sin \theta \cos^2 \theta = 0 .$$

30. If we convert to polar coordinates we obtain

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + xy + y^2}{x^2 + y^2} = \lim_{r \to 0} \frac{r^2 + (r \cos \theta) \cdot (r \sin \theta)}{r^2} = 1 + \cos \theta \cdot \sin \theta .$$

Since the expression in the right hand limit is independent of r and varies between $\frac{1}{2}$ and $\frac{3}{2}$ as θ ranges over all real numbers, it follows that there is no limit value in this case and hence the limit does not exist.

36. The only possible source of trouble would be a zero in the denominator, but we know that $x^2 + 1 > 0$ and hence g is always continuous.

40. Away from the origin, the function is a quotient of two polynomials such that the denominator is never zero, and hence the function is continuous except possibly at the origin. Therefore it is only necessary to see what happens at the origin. But we have

$$\lim_{(x,y)\to(0,0)} \frac{x^3 + x^2 + xy^2 + y^2}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} \frac{(x^2 + y^2)(x + 1)}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} x + 1 = 1$$

so that the limit exists, but on the other hand we have

$$\lim_{(x,y)\to(0,0)} g(x, y) = 1$$

while $g(0, 0) = 2$, and therefore g is not continuous at the origin.

42. If $(x, y) \neq (0, 0)$ then we may simplify the given expression as follows:

$$\frac{x^3 + xy^2 + 2x^2 + 2y^2}{x^2 + y^2} = \frac{(x^2 + y^2)(x + 2)}{x^2 + y^2} = x + 2$$

so that $c = 2$ and the function $g(x, y)$ is seen to be equivalent to $x + 2$.

44. This function is equivalent to $\mathbf{f}(x, y, z) = (-5y, 5x - 6z, 6y)$. Since each of the component functions from $\mathbf{R}^3$ to $\mathbf{R}$ is continuous, so is $\mathbf{f}$.