Some items from the lectures on Colley, Section 2.3

PARTIAL DIFFERENTIATION AND LINEAR APPROXIMATIONS. The text gives an example of a function $f(x, y)$ which is defined by a quotient of two polynomials everywhere except at the origin, and even has partial derivatives with respect to x and y at the origin, but is not continuous there. In contrast, for functions of one variable, a function which is differentiable at a point is automatically continuous there. However, if a function $f(x_1, x_2, \cdots)$ has continuous first partial derivatives at $(a_1, a_2, \cdots)$, then it is continuous there, and in fact one can write

$$f(\mathbf{a}+\mathbf{h}) \quad = \quad f(\mathbf{a}) \ + \ \nabla f(\mathbf{a}) \cdot \mathbf{h} \ + \ |\mathbf{h}|\, T(\mathbf{h})$$

where the "trash term" $T(\mathbf{h})$ goes to zero as $\mathbf{h} \to \mathbf{0}$. — We shall illustrate these points with the following example:

$$f(x, y) \quad = \quad \frac{1}{x^2 + y^2}, \qquad \mathbf{a} \quad = \quad (1, 1)$$

Notice that $f(1, 1) = \frac{1}{2}$. It will be helpful to let Δx, Δy and Δf denote the differences between the values of x, y and f if the point (x, y) takes values other than $(1, 1)$. In this case, we have the approximation formula

$$f(x, y) \quad \approx \quad f(1, 1) \ + \ \frac{\partial f}{\partial x}(1, 1) \cdot \Delta x \ + \ \frac{\partial f}{\partial y}(1, 1) \cdot \Delta y$$

and since

$$\nabla f \quad = \quad \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \quad = \quad \left(\frac{-2x}{(x^2 + y^2)^2}, \frac{-2y}{(x^2 + y^2)^2} \right)$$

we see that $\nabla f(1, 1) = (-\frac{1}{2}, -\frac{1}{2})$, so that we obtain the following linear approximation formula for (x, y) close to $(1, 1)$:

$$f(x, y) \quad \approx \quad \frac{1}{2} \ + \ \left(-\frac{1}{2} \right) \Delta x \ + \ \left(-\frac{1}{2} \right) \Delta y$$

Suppose first that we take $(x, y) = (0.95, 1.05)$, so that $\Delta x = -0.05$ and $\Delta y = 0.05$. In this case the approximation formula yields a value of $\frac{1}{2}$, and the actual value of the function is 0.5216467..., which means that the approximation is within 4.5 percent of the actual value. If we take $(x, y) = (1.02, 1.03)$, then $\Delta x = 0.02$ and $\Delta y = 0.03$, so that the approximation formula yields an approximate value of 0.475; in this case the actual value is 0.4758949..., so the approximation is accurate to within 0.2 percent. As a rule, the linear approximations become more accurate fairly quickly as one gets closer and closer to $\mathbf{a}$.

PARTIAL DIFFERENTIATION AND ERROR ESTIMATES. Partial derivatives are often useful in analyzing experimental errors. We shall illustrate this with a simple example involving electrical circuits. The starting point is the basic formula

$$P \quad = \quad I^2 \cdot R$$

where P denotes the power (measured in watts), I denotes the current (measured in amperes) and R denotes the resistance (measured in ohms). Suppose now that we can measure the current and resistance, so that the formula yields the power that the circuit uses. Measurements are imperfect, so suppose that our measurements for the power and resistance are accurate to within 2 percent. We would like to estimate the maximum error in our computation of the power usage.

In order to continue, we need to give specific values for the current and resistance, so let us assume that the measured current is 4 amperes and the measured resistance is 30 ohms, so that the predicted power usage is 480 watts. If we again let ΔX denote the error (or change) in the value of the quantity X, then the basic estimate from Section 2.3 in the text implies that

$$\Delta P \quad \approx \quad \frac{\partial P}{\partial I} \cdot \Delta I \ + \ \frac{\partial P}{\partial R} \cdot \Delta R \quad =$$

$$2\, I\, R\, \Delta I \ + \ I^2\, \Delta R \ .$$

This means that the magnitude $|\Delta P|$ of the error in the calculated value for the power usage should be at most the sum of the absolute values of $2\, I\, R\, \Delta I$ and $I^2\, \Delta R$. In other words, we have the following:

$$|\Delta P| \ \text{ is roughly } \quad \leq \quad 2\, I\, R\, |\Delta I| \ + \ I^2\, |\Delta R|$$

We know that $I = 4$ and $R = 30$, and since the accuracy in measuring these quantities is 2 percent, it follows that $|\Delta I| \leq 0.08$ and $|\Delta R| \leq 0.6$. Substituting these into the error estimate, we see that a rough upper bound for $|\Delta P|$ is given by

$$(2 \cdot 4 \cdot 30 \cdot 0.08) \ + \ (16 \cdot 0.6) \quad = \quad 19.2 \ + \ 9.6 \quad = \quad 28.8 \text{ watts.}$$

Since $28.8/480 = 0.06$, this means that the computed value for the power usage is accurate to within about 6 percent.

Error estimates of this sort play a crucial role in checking whether experimental data is compatible with theoretical models, and even within mathematics itself the analysis of error terms is often extremely important.

TANGENT PLANES FOR FAMILIAR SURFACES. It may be helpful to check that the textbook definition of tangent plane corresponds to the intuitive notions of tangent planes for an ordinary plane and a sphere. In the case of a plane, assume it is given by an equation of the form $z = f(x, y) = ax + by + c$ and that $(p, q, ap + bq + c)$ is a point on this plane. We then have

$$f(x, y) - f(p, q) \quad = \quad (ax + by + c) \ - \ (ap + bq + c) \quad = \quad a(x - p) \ + \ b(y - q) \ .$$

Since $\nabla f(x, y) = (a, b)$, the equation for the tangent plane at (p, q, r) — where $r = f(p, q)$ – is given by

$$f(x, y) \quad = \quad f(p, q) \ + \ \nabla f(p, q) \cdot (x - p, y - q)$$

and direct calculuation shows that this is equivalent to the previously displayed equation.

Now consider the northern hemisphere of the sphere $x^2 + y^2 + z^2 = 1$, which is given by

$$z \quad = \quad \sqrt{1 - x^2 - y^2}, \qquad \text{where} \quad x^2 + y^2 \quad < \quad 1 \ .$$

Direct calculation shows that the normal (or perpendicular direction) to the tangent plane is

$$\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1 \right) \quad = \quad \left(-\frac{x}{z}, -\frac{y}{z}, -1 \right)$$

and since $z > 0$ by the restrictions on x and y, this vector is a nonzero multiple of (x, y, z), which corresponds to the classical Greek description of the tangent plane to a sphere as the plane which is perpendicular to the radius at the point of contact.

SOME ONLINE REFERENCES. Clickable links for a few YouTube-like videos are given in the file `weblinks23.pdf` which as usual can be found in the course directory.

Answers to selected exercises from Colley, Section 2.3

4. The partial derivatives are given as follows:

$$\frac{\partial f}{\partial x} = \frac{(1+x^2+3y^4)(3x^2) - (x^2-y^2)(2x)}{(1+x^2+3y^4)^2}$$

$$\frac{\partial f}{\partial y} = \frac{(1+x^2+3y^4)(-2y) - (x^2-y^2)(12y^3)}{(1+x^2+3y^4)^2}$$

10. The partial derivatives are given as follows:

$$\frac{\partial F}{\partial x} = a\,e^{ax}\cos by + b\,e^{az}\cos bx, \quad \frac{\partial F}{\partial y} = -b^{\,ax}\sin by, \quad \frac{\partial F}{\partial z} = a\,e^{az}\sin bx$$

12. The partial derivatives are given as follows:

$$\frac{\partial F}{\partial x} = \cos x^2y^3z^4 \cdot 2\,x\,y^3\,z^4, \quad \frac{\partial F}{\partial y} = \cos x^2y^3z^4 \cdot 3\,x^2\,y^2\,z^4, \quad \frac{\partial F}{\partial z} = \cos x^2y^3z^4 \cdot 4\,x^2\,y^3\,z^3$$

17. The definitions of $f(x,y,z)$ and the gradient imply that

$$\nabla f(x,y,z) = (y - \sin yz, x + \cos z - xz\cos yz, -y\sin z - xy\cos yz)\ .$$

If we evaluate this at $(2,-1,\pi)$, we see that

$$\nabla f(2,-1,\pi) = (-1-\sin(-\pi), 2+\cos\pi - 2\pi\cos(-\pi), \sin\pi + 2\cos(-\pi)\,) = (-1, 1+2\pi, -2)\ .$$

18. We have

$$\nabla f(x,y) = \left(\frac{ye^{xy}+1}{x-y}, \frac{xe^{xy}-1}{x-y}\right)$$

which means that $\nabla f(2,1) = (e^2+1, 2e^2-1)$.

19. We have $\nabla f(x,y,z) = (e^{-z}, e^{-z}, -(x+y)e^{-z})$, so that $\nabla f(3,-1,0) = (1,1,-2)$.

20. In this problem and the next one, the rows of the matrix are given by the coordinates of the gradients of the respective coordinate functions (so the first row is essentially the gradient of the first coordinate function, *etc.*). Since there is only one coordinate in the problem, we have $df = \nabla f$. Now

$$\nabla f(x,y) = \left(\frac{1}{y}, \frac{-x}{y^2}\right)$$

and therefore we have $Df(3,2) = (1/2, -3/4)$.

24. Recall the first sentence in the answer to problem 20. If we apply this principle, we see that

$$D\mathbf{f}(x,y) \quad = \quad \begin{bmatrix} 2xy & x^2 \\ 1 & 2y \\ -\pi y \sin \pi xy & -\pi x \sin \pi xy \end{bmatrix}$$

and if we evaluate at $(2,-1)$ we obtain the following:

$$D\mathbf{f}(2,-1) \quad = \quad \begin{bmatrix} -4 & 4 \\ 1 & -2 \\ 0 & 0 \end{bmatrix}$$

27. The function is differentiable because it is a quotient of two polynomials which is defined whenever $(x,y,z) \neq (0,0,0)$; the three partial derivatives of such a quotient will also be quotients of polynomials which are defined if $(x,y,z) \neq (0,0,0)$, and therefore they are all continuous.

28. The coordinate functions are built from polynomial functions using the four standard arithmetic operations of addition, subtraction, multiplication and division, and therefore they have continuous partials provided none of the denominators in question are zero.

29. (a) The function on the right hand side has continuous partial derivatives at $(-1,0,0)$.

(b) In order to streamline the formulas we shall let f_x and f_y denote the partial derivatives with respect to x and y respectively. The equation for the tangent plane then is given as follows:

$$z \quad = \quad f(-1,0) \ + \ f_x(-1,0)\big(x-(-1)\big) \ + \ f_y(-1,0)(y-0)$$

Now $f_x(x,y) = 3x^2 - 7y$ and $f_y(x,y) = -7x + e^y$, so that $f_x(-1,0) = 3$ and $f_y(-1,0) = 8$. Thus the equation of the plane is $z = 3(x+1) + 8y = 3x + 8y + 3$.

30. In this problem we have $f_x(x,y) = -4y \sin xy$ and $f_y(x,y) = -4x \sin xy$. If we substitute the values $(x,y,z) = (\pi/3, 1, 2)$ into the equation for the tangent plane, we conclude that

$$z \quad = \quad 2 \ - \ 2\sqrt{3}\left(x \ - \ \frac{\pi}{3}\right) \ - \ \left(\frac{2\pi}{\sqrt{3}(y-1)}\right) \ .$$

32. The two partial derivatives are $f_x(x,y) = 2x - 6$ and $f_y(x,y) = 3y^2$, so the normal direction to the tangent plane is $(2x-6, 3y^2, -1)$. In order for the tangent plane to be equal or parallel to the plane $4x - 12y + z = 7$ it is necessary for the normal direction of the tangent plane to be a nonzero multiple of the normal direction for the given plane. In other words, we need to solve the equation

$$(2x-6, 3y^2, -1) \quad = \quad (4k, -12k, k)$$

for some $k \neq 0$. Inspection of the third coordinates shows that this can only happen if $k = -1$, so we need to solve the equations $2x - 6 = -4$ and $3y^2 = 12$. The solutions turn out to be $(x,y) = (1, \pm 2)$. To finish the problem, one must find the tangent planes to the surface for the points associated to these two values of (x,y). The equation for one is $4x - 12y + z = -17$, and the equation for the other is $4x - 12y + z = 15$.

35. Here the partial derivatives are $f_x(x,y) = e^x + y = f_y(x,y)$. In this case both the linear approximation and the actual value of the function are equal to 1.

36. The partial derivatives are

$$\frac{\partial f}{\partial x} = -\pi y \sin \pi x y, \qquad \frac{\partial f}{\partial x} = -\pi x \sin \pi x y$$

and if we use this to describe the linear approximation at $(x, y) = (.98, .51)$ we obtain the approximate value $f(.98, .51) \approx 3$. The actual value is given up to 8 decimal places by $f(.98, .51) = 3.00062832$.

50. Students will not be responsible for knowing how to work this problem.