## Some items from the lectures on Colley, Section 2.4

**RESTRCITION OF COVERAGE.** The material on the Newton-Raphson method for finding approximate solutions to equations of the form $f(x_1, \cdots) = C$ will **NOT** be covered in this course.

**THIRD ORDER PARTIAL DERIVATIVES.** The equality of mixed order second order partial derivatives $f_{uv} = f_{vu}$ yields equations for various mixed partial derivatives of higher order (see page 129 of the text). We shall enumerate the distinct possibilities for third order partial derivatives and show that there are ten of them.

The crucial point is that a third order partial derivative of a function of $x$, $y$ and $z$ only depends upon the numbers of $\partial x$, $\partial y$ and $\partial z$ terms in the denominator. Here are the possibilities.

**(A)** *One of these terms appears three times, and neither of the others appears at all.* There are three possibilities given by $f_{xxx}$, $f_{yyy}$, and $f_{zzz}$.

**(B)** *One of these terms appears two times, one appears one time, and the remaining one does not appear.* There are six possibilities given by $f_{xxy}$, $f_{yyx}$, $f_{yyz}$, $f_{zzy}$, $f_{xxz}$, $f_{zzx}$, and rearrangements of the indices.

**(C)** *Each of these terms appears exactly once.* There is only one possibility given by $f_{xyz}$ (and rearrangements of these terms).

## Answers to selected exercises from Colley, Section 2.4

**9.** We have $f_x(x,y) = 3x^2y^7 + 3y^2 - 7y$ and $f_y(x,y) = 7x^3y^6 + 6xy - 7x$, which means that second order partials are given as follows:

$$f_{xx}(x,y) = 6xy^7, \qquad f_{yy}(x,y) = 42x^3y^5 + 6x, \qquad f_{xy}(x,y) = f_{yx}(x,y) = 21x^2y^6 + 6y$$

**11.** The first order partial derivatives are

$$f_x(x,y) = \frac{-y}{x^2}\, e^{y/x} + ye^{-x}, \qquad f_y(x,y) = \frac{1}{x}\, e^{y/x} - e^{-x}$$

and the second order partial derivatives are given as follows:

$$f_{xx}(x,y) = \frac{2y}{x^3}\, e^{y/x} + \frac{y^2}{x^4}\, e^{y/x} - ye^{-x}$$

$$f_{xy}(x,y) = f_{yx}(x,y) = \frac{-1}{x^2}\, e^{y/x} - \frac{y}{x^3}\, e^{y/x} + e^{-x}$$

$$f_{yy}(x,y) = \frac{1}{x^2}\, e^{y/x}$$

**13.** The first order partial derivatives are

$$f_x(x,y) = \frac{-2\sin x\cos x}{(\sin^2 x + 2e^y)^2} = \frac{-\sin 2x}{(\sin^2 x + 2e^y)^2}, \qquad f_y(x,y) = \frac{-2e^y}{(\sin^2 x + 2e^y)^2}$$

and the second order partial derivatives are given as follows:

$$f_{xx}(x,y) = \frac{(\sin 2x + 2e^y)(-2\cos 2x) + 2\sin^2 x}{(\sin 2x + 2e^y)^3}$$

$$f_{xy}(x,y) = f_{yx}(x,y) = \frac{4e^y \sin 2x}{(\sin 2x + 2e^y)^3}$$

$$f_{yy}(x,y) = \frac{2e^y(2e^y - \sin^2 x)}{(\sin 2x + 2e^y)^3}$$

**14.** The first order partial derivatives are

$$f_x(x,y) = 2xe^{x^2+y^2}, \qquad f_y(x,y) = 2ye^{x^2+y^2}$$

and the second order partial derivatives are given as follows:

$$f_{xx}(x,y) = 2e^{x^2+y^2} + 2x \cdot 2xe^{x^2+y^2} = e^{x^2+y^2}(2+4x^2)$$

$$f_{xy}(x,y) = f_{yx}(x,y) = 4xye^{x^2+y^2}, \qquad f_{yy}(x,y) = e^{x^2+y^2}(2+4y^2)$$

**16.** The first order partial derivatives are

$$f_x(x,y,z) = yze^{xyz}, \qquad f_y(x,y,z) = xze^{xyz}, \qquad f_z(x,y,z) = xye^{xyz}$$

and the second order partial derivatives are given as follows:

$$f_{xx}(x,y,z) = y^2z^2e^{xyz}, \qquad f_{yy}(x,y,z) = x^2z^2e^{xyz}, \qquad f_{zz}(x,y,z) = x^2y^2e^{xyz}$$

$$f_{xy}(x,y,z) = f_{yx}(x,y,z) = ze^{xyz}(1+xyz)$$

$$f_{xz}(x,y,z) = f_{zx}(x,y,z) = ye^{xyz}(1+xyz)$$

$$f_{yz}(x,y,z) = f_{zy}(x,y,z) = xe^{xyz}(1+xyz)$$

**18.** The first order partial derivatives are

$$F_x(x,y,z) = 6x^2y + z^2 - 7yz$$

$$F_y(x,y,z) = 2x^3 + 3y^2z^5 - 7xz$$

$$F_z(x,y,z) = 2xz + 5y^3z^4 - 7xy$$

and this yields the following equations for the higher order partial derivatives:

**(a)** The straight second order partial derivatives are $F_{xx}(x,y,z) = 12xy$, $F_{yy}(x,y,z) = 6yz^5$, and $F_{zz}(x,y,z) = 20y^3z^3 + 2x$.

**(b)** The mixed second order partial derivatives are $F_{xy}(x,y,z) = 6x^2 - 7z = F_{yx}(x,y,z)$, $F_{xz}(x,y,z) = 2z - 7y = F_{zx}(x,y,z)$, and $F_{yz}(x,y,z) = 15y^2z^4 - 7x = F_{zy}(x,y,z)$.

**(c)** The third order partial derivatives are given by $F_{xyx}(x,y,z) = 12x = F_{xxy}(x,y,z)$. We knew that these would be equal because they are the mixed partials of $F$ (specifically, $(F_x)_{yx} = (F_x)_{xy}$ ).

**(d)** $F_{xyz}(x,y,z) = -7 = F_{yzx}(x,y,z)$.

**20. (a)** For the first function, the second partial derivatives with respect to $x$ and $y$ are both (the function with constant value) 2 and the second partial derivative with respect to $z$ is equal to $-4$; if we add these up we obtain $f_{xx} + f_{yy} + f_{zz} = 2 + 2 - 4 = 0$ so that f is harmonic. For the second function, the respective second partial derivatives are 2, $-2$ and 2 respectively, so that $f_{xx} + f_{yy} + f_{zz} = -2 \neq 0$, which means that $f$ is **NOT** harmonic.