Some items from the lectures on Colley, Section 2.5

CHAIN RULE FORMULAS. There are several formulas in the text, and the best way to remember them is in sequence. First there is the version for a composite function $y = f(u_1, \cdots)$ where $u_i = u_i(t)$. This is like the ordinary chain rule but it is a sum of terms, one for each intermediate variable u_i. Since partial derivatives may be viewed as special types of ordinary derivatives, one can use the formula for $u_i = u_i(t)$ to recover the formula when each u_i is a function of several variables, say v_j. Finally, for the version involving vector valued function one merely needs to remember that the derivative matrix has rows given by the gradients of the respective coordinate functions; for example if f_i denotes the i^{th} coordinate function of $\mathbf{f}$, then ∇f_i is the i^{th} row of $D\mathbf{f}$. We can then apply the chain rule for scalar valued functions to retrieve the formula in Theorem 5.3, on page 148 of the text. In this course, it is very unlikely that there will be problems involving the chain rule in the most complicated setting described above.

Example. Suppose we are given a function $g(u, v) = f(u/v, u^2v)$. Find the partial derivative of g with respect to u.

To start, let $s = u/v$ and $t = u^2v$. Then the chain rule implies that

$$\frac{\partial g}{\partial u} \quad = \quad \frac{\partial f}{\partial s} \cdot \frac{\partial s}{\partial u} \; + \; \frac{\partial f}{\partial t} \cdot \frac{\partial t}{\partial u}$$

and if we express s and t in terms of u and v this reduces to

$$\frac{\partial f}{\partial s} \cdot \frac{1}{v} \; + \; \frac{\partial f}{\partial t} \cdot 2uv$$

so that the partial derivative is given by the dot product formula

$$\frac{\partial g}{\partial u} \quad = \quad \nabla f(u/v, u^2v) \cdot (1/v, 2uv) \ .$$

One can compute the partial derivative of g with respect to v in a similar manner (do it!).

Answers to selected exercises from Colley, Section 2.5

1. We shall work this using the Chain Rule:

$$\frac{df}{dt} \quad = \quad \frac{\partial f}{\partial x}\frac{dx}{dt} \; + \; \frac{\partial f}{\partial y}\frac{dy}{dt} \; + \; \frac{\partial f}{\partial z}\frac{dz}{dt} \quad =$$

$$(2x + yz)(6) \; + \; (-3y^2 + xz)(2\cos 2t) \; + \; (xy)(2t) \quad =$$

$$[2(6t + 7) \; + \; (\sin 2t)(t^2)](6) \; + \; [-3\sin^2 2t \; + \; (6t + 7)t^2](2\cos 2t) \; + \; [(6t + 7)\sin 2t](2t)$$

This can be rewritten (simplified), but we shall not do so; an answer in the given form would receive full credit on an exam.

2. We shall also work this using the Chain Rule:

$$\frac{\partial f}{\partial s} \quad = \quad \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \quad = \quad y\cos(xy) + x\cos(xy)2s \quad = \quad \cos[(s+t)(s^2+t^2)][(s^2+t^2)+(s+t)(2s)]$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = y\cos(xy) + x\cos(xy)2t = \cos[(s+t)(s^2+t^2)][(s^2+t^2)+(s+t)(2t)]$$

3. (a) We want

$$\frac{dP}{dt} = \frac{\partial P}{\partial x}\frac{dx}{dt} + \frac{\partial P}{\partial y}\frac{dy}{dt} + \frac{\partial P}{\partial z}\frac{dz}{dt} =$$

$$\frac{12xz}{y}(-2\sin t) - \frac{6x^2z}{y^2}(2\cos t) + \frac{6x^2}{y}(3) =$$

$$\frac{12(2\cos t)(3t)}{(2\sin t)}(-2\sin t) - \frac{6(2\cos t)^2(3t)}{(2\sin t)^2}(2\cos t) + \frac{6(2\cos t)^2}{(2\sin t)}(3) =$$

$$-72t\cos t - 36t\frac{\cos^3 t}{\sin^2 t} + 6\frac{\cos^2 t}{\sin t}$$

Evaluating this at $t = \pi/4$, we see that the numerical value for $P'(\pi/4)$ is equal to

$$\frac{36 - 27\pi}{\sqrt{2}} .$$

(b) We are only going to compute the derivative by the Chain Rule.

(c) The linear approximation formula implies $\Delta P \approx P'(\pi/4)\Delta t \approx -.34523$, and if we substitute the value for the first factor we conclude that

$$P(\pi/4 + .01) \approx P(\pi/4 + \Delta P) \approx \frac{9\pi}{\sqrt{2}} - .34523 = 19.6477 .$$

4. Although we are not given an explicit description of y in terms of s and t, we still know enough to compute the partial derivative $z_t(2,1)$ because we know that $y_t(2,1) = 0$. The chain rule implies that

$$z_t(2,1) = z_x\, x_t(2,1) + z_y\, y_t(2,1) = z_x\, x_t(2,1) .$$

Now $z_x = 2x = 2st$ (since $x = st$) and $x_t = s$, so the right hand side reduces to the value of $2s^2t$ at $(2,1)$. This is equal to 8.

5. Here $V = LWH$, so

$$V' = L'WH + W'LH + H'LW$$

and if we substitute the values $(L,V,W) = (7,5,4)$ and $(L',W',H') = (0.75, 0.5, -1)$ we find that $V' = -6$ cubic inches per minute; since this rate is negative, the volume of the dough is decreasing at noon.

6. If the length of the butter stick ℓ and the length of an edge of the cross section is e, then the volume $V = x^2y$ and the rate at which the volume is changing is

$$\frac{dV}{dt} = 2xy\frac{dx}{dt} + x^2\frac{dy}{dt} = 2(1.5)(6)(-.125) + (1.5)^2(-.25) = -2.8125 \text{ in}^3/\text{min} .$$

10. We first calculate the factors on the left side:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = z_u \cdot 1 + z_v \cdot 1 \qquad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = z_u \cdot 1 - z_v \cdot 1$$

When we multiply these two expressions together, we obtain $(z_u)^2 - (z_v)^2$.

11. We begin by finding the partial derivatives of u:

$$\frac{\partial u}{\partial x} \;=\; y\frac{x^2-y^2}{(x^2+y^2)^2}\,, \qquad \frac{\partial u}{\partial y} \;=\; x\frac{y^2-x^2}{(x^2+y^2)^2}$$

Now $w = f(u)$, so by the Chain Rule we have

$$x\,w_x \;+\; y\,w_y \;=\; x\,u_x\,f' \;+\; y\,u_y\,f'$$

and if we substitute the previously derived expressions for u_x and u_y we obtain something that simplifies to 0.

12. We begin by finding the partial derivatives of u:

$$\frac{\partial u}{\partial x} \;=\; \frac{4xy^2}{(x^2+y^2)^2}\,, \qquad \frac{\partial u}{\partial y} \;=\; \frac{-4x^2y}{(x^2+y^2)^2}$$

Now $w = f(u)$, so by the Chain Rule we have

$$x\,w_x \;+\; y\,w_y \;=\; x\,f'\,u_x \;+\; y\,f'\,u_y$$

and if we substitute the previously derived expressions for u_x and u_y we obtain something that simplifies to 0.

15. Students are not responsible for knowing how to work this problem.

16. Ditto.

19. Ditto.

26.(a) We are assuming that $F(\,x, y(x)\,) = 0$. Applying the Chain Rule we find that

$$0 \;=\; \frac{d}{dx}F\big(\,x, y(x)\,\big) \;=\; F_x\,x' \;+\; F_y\,y' \;=\; 0$$

and if we recall that $x' = 1$ and $F_y \neq 0$ then it follows that we may solve this equation for y' and find that

$$y' \;=\; -\frac{F_x}{F_y}\;.$$

(b) In this case the formula from (a) yields $y' = (3x^2)/(2y)$. If we solve for y we obtain the equation $y = x^{3/2}$, so that $y' = \frac{3}{2}x^{1/2}$. On the other hand, if we substitute the explicit description of y into the first equation above, then the latter simplifies to $\frac{3}{2}x^{1/2}$.

29. *Hint:* Use the equations from Exercise 28(a) for $F(x,y,z) = x^3z + y\cos z + (\sin y)/z = 0$. Specifically, $z_x = -(F_x/F_z)$ and $z_y = -(F_y/F_z)$.