Some items from the lectures on Colley, Section 2.6

RESTRICTION OF COVERAGE. The material on the general case of the Inverse Function Theorem will **NOT** be covered in this course. However, the Implicit Function Theorem for scalar valued functions **HAS BEEN COVERED.** This goes up to, but not including, the last paragraph on page 164 of the text (which begins with "As just mentioned ...").

Example. Suppose that y is given implicitly as a function of x by the equation $x^3+y^3-9=0$ and we know that $y(1)=2$. Find $y'(1)$ without solving for y' explicitly.

In this case the defining equation $F(x,y)=x^3+y^3-9$, and the chain rule formula for implicit differentiation implies that

$$y' \quad = \quad -\frac{F_x}{F_y}$$

where F_w is the partial derivative with respect to $w=x,y$. In order to use this formula, we need to know that $F_y(1,2)$ is nonzero, but since $\nabla F=(3x^2,3y^2)$ this condition is clearly satisfied. Now the value of the gradient $\nabla F(1,2)=(3,12)$, and if we now substitute these values for F_x and F_y we find that $y'=-\frac{1}{4}$.

THIRD ORDER PARTIAL DERIVATIVES.

Answers to selected exercises from Colley, Section 2.6

5. $\nabla f(x,y)=(e^x-2xy,-x^2)$, so $\nabla f(1,2)=(e-4,-1)$, and

$$D_{\mathbf{u}}f(\mathbf{a}) \quad = \quad (e-4,-1)\cdot\frac{1}{\sqrt{5}}(2,1) \quad = \quad \frac{1}{\sqrt{5}}(2e-9)\ .$$

6. $\nabla f(x,y,z)=(yz,xz,xy)$, so $\nabla f(-1,0,2)=(0,-2,0)$ and

$$D_{\mathbf{u}}f(\mathbf{a}) \quad = \quad (0,-2,0)\cdot\frac{1}{\sqrt{5}}(-1,0,2) \quad = \quad 0\ .$$

7. $\nabla f(x,y,z)=-e^{x^+Cy^2+z^2}(2x,2y,2z)$, so $\nabla f(1,2,3)=-e^{-14}(2,4,6)$ and

$$D_{\mathbf{u}}f(\mathbf{a}) \quad = \quad -e^{-14}(2,4,6)\cdot\frac{1}{\sqrt{3}}(1,1,1) \quad = \quad -4\sqrt{3}\,e^{-14}\ .$$

11. The gradient direction for the function h is $\nabla h=(-6xy^2,-6x^2y)$. The most rapid decrease occurs in the direction $\nabla h(1,-2)=(-24,12)$, and the direction in which the depth remains constant is given by a unit vector which is a multiple of $(1,2)$ (to get a vector of unit length, divide by $\sqrt{5}$).

12. The gradient is $\nabla f(3,7)=(3,-2)$.

(a) To warm up we head in the direction of the gradient; this is the unit vector $(3,-2)/\sqrt{13}$.

(b) To cool off we head in the opposite direction; this is the unit vector $(-3,2)/\sqrt{13}$.

(c) To maintain temperature we head in a direction perpeudicular to the gradient; namely, $(2,3)/\sqrt{13}$.

17. $\nabla f(x,y,z) = (-ze^y \sin x, ze^y \cos x, e^y \cos x)$ and $\nabla f(\pi,0,-1) = (0,1,-1)$. So the equation of the tangent plane is $0 = (0,1,-1)\cdot(x-\pi, y, z+1)$ or $y - z = 1$.

17. $\nabla f(x,y,z) = (2y^2+yz, 4xy+xz, xy-4z)$ and $\nabla f(2,-3,3) = (9,-18,-18)$. Therefore the equation of the tangent plane is $0 = (9,-18,-18)\cdot(x-2, y+3, z-3)$ or equivalently $x - 2y - 2z = 2$.

20. We shall only work this problem using the gradient formula for the normal to the surface. This gradient is given by $\nabla f(x,y,z) = (2x+5z, -4y, 5x)$, so $\nabla f(-1,0,-6/5) = (-8,0,-5$, and therefore the equation for the tangent plane is $0 = (-8,0,-5)\cdot(x+1, y, z+6/5)$, or equivalently $-8x - 5z = 14$.

22. The gradient of f at (x_0,y_0,z_0) is $(3x_0^2, -4y_0, 2z_0)$. For this to be perpendicular to the given line, $(3x_0^2, -4y_0, 2z_0)$ must equal $k(3,2,-\sqrt{2})$ for some nonzero constant k. This means that $x_0^2 = -2y_0$ and $z_0 = -(\sqrt{2}/2)x_0^2$. Substituting this back into the equation of the surface, we get that $x_0^3 - 2x_0^4/4 + x_0^4/2 = 27$ or $x_0 = 3$. Therefore the point in question is $(3, -9/2, -9\sqrt{2}/2)$.

23. The tangent plane to the surface at a point (x_0,y_0,z_0) is

$$0 \;=\; 18x_0(x-x_0) \;-\; 90y_0(y-y_0) \;+\; 10z_0(z-z_0)\;.$$

For this to be parallel (or equal) to the plane with equation $x + 5y - 2z = 7$, the gradient vector $(18x_0, -90y_0, 10z_0)$ must be equal to $k(1,5,-2)$ for some nonzero constant k. This means that $y_0 = -x_0$ and $z_0 = (-18/5)x_0$. If we substitute these back into the equation of the hyperboloid with equation $9x^2 - 45y^2 + 5z^2 = 45$ we see that

$$45 \;=\; 9x_0^2 \;-\; 45x_0^2 \;+\; 5(18^2/5^2)x_0^2$$

and if we solve for x_0 we find that $x_0 = \pm 5/4$. This means that the points are $(5/4, -5/4, -9/2)$ and $(-5/4, 5/4, 9/2)$.

26.(a) S is the level set at height 0 for $f(x,y,z) = x^2 + 4y^2 - z^2$, and $\nabla f(3,-2,-5) = (6,-16,10)$. Thus the equation of the tangent plane is given by

$$6(x-3) \;-\; 16(y+2) \;+\; 10(z+5) \;=\; 0$$

or equivalently $3x - 8y + 5z = 0$.

(b) The gradient of f at $(0,0,0)$ is $(0,0,0)$, so the gradient cannot be used as a normal vector. For our purposes, this will be enough to say that the surface does not have a reasonable tangent plane at the origin.

27.(a) We know that $\nabla f(x,y,z) = (3x^2 - 2xy^2, -2x^2y, 2z)$, so that $\nabla f(2,-3/2,1) - (3,12,2)$. Thus the equation of the tangent plane is

$$3(x-2) \;+\; 12(y+3/2) \;+\; (z-1) \;=\; 0$$

or equivalently $3x + 12y + 2z + 10 = 0$.

(b) $\nabla f(0,0,0) = (0,0,0)$, so the gradient cannot be used as a normal vector. For our purposes, this will be enough to say that the surface does not have a reasonable tangent plane at the origin.

31. $\nabla f(5,-4) = (10,8)$, so the equations of the normal line are $x(t) = 10t + 5$ and $y(t) = 8t - 4$ or equivalently $8x - 10y = 80$.

32. $\nabla f(-1,\sqrt{2}) = (5, 2\sqrt{2})$, so the equations of the normal line are $x(t) = 5t - 1$ and $y(t) = 2\sqrt{2}t - \sqrt{2}$ or equivalently $2\sqrt{2}x - 5y = -7\sqrt{2}$.