Some items from the lectures on Colley, Section 3.1

RESTRICTION OF COVERAGE. The material on Kepler's Laws will **NOT** be covered in this course.

Answers to selected exercises from Colley, Section 3.1

8. The velocity is given by $\mathbf{v}(t) = \mathbf{x}'(t) = (-5\sin t, 3\cos t)$, the speed by $|\mathbf{v}'|(t) = (-5\sin t)^2 + (3\cos t)^2 = \sqrt{9 + 16\sin^2 t}$, and the acceleration by $\mathbf{a}(t) = \mathbf{x}''(t) = (-5\cos t, -3\sin t) = -\mathbf{x}(t)$.

10. The velocity is given by $\mathbf{v}(t) = \mathbf{x}'(t) = (e^t, 2e^{2t}, 2e^t)$, the speed by

$$|\mathbf{v}'|(t) \quad = \quad \sqrt{5e^{2t} + 4e^{4t}} \quad = \quad e^t\sqrt{5 + 4e^{2t}}$$

and the acceleration by $\mathbf{a}(t) = \mathbf{x}''(t) = (e^t, 4e^{2t}, 2e^t)$.

16. We have $\mathbf{x}(\pi/3) = (2, -3\sqrt{3}/2, 5\pi/3)$ and $\mathbf{x}'(t) = (-4\sin t, -3\cos t, 5)$ so that $\mathbf{x}'(\pi/3) = (-2\sqrt{3}, -3/2, 5)$. Hence the equation of the tangent line at $t = \pi/3$ is $\mathbf{L}(t) = (2, -3\sqrt{3}/2, 5\pi/3) + (-2\sqrt{3}, -3/2, 5) \cdot (t - \pi/3)$.

18. We have $\mathbf{x}(1) = (\cos e, 2, 1)$ and $\mathbf{x}'(t) = (-e^t \sin e^t, -2t, 1$ so that $\mathbf{x}'(-e\sin e, -2, 1)$. Hence the equation of the tangent line at $t = 1$ is $\mathbf{L}(t) = (\cos e, 2, 1) + (-e\sin e, -2, 1)(t-1) = (\cos e + e\sin e - (e\sin e)t, 4 - 2t, t)$.

26.(a) In order to determine the time(s) at which the balls meet, we mmust set $\mathbf{x}(t) = \mathbf{y}(t)$ and solve for t:

$$\left(t^2 - 2, \tfrac{1}{2}\,t^2 - 1\right) \quad = \quad (t, 5 - t^2)\ .$$

Comparing first coordinates, we have $t^2 - 2 = t$, which is equivalent to $t^2 - t - 2 = 0$, which in turn implies that $t = -1, 2$. This means that the first coordinates for the positions of the balls are the same at these two values; we need to check which, if any, of these two values yield the same second coordinates. Since $\mathbf{x}(-1) = (-1, -1)$ and $\mathbf{y}(-1) = (-1, 4)$, the value $t = -1$ does not yield a collision point. However, we do have $\mathbf{x}(2) = (2, 1) = \mathbf{y}(2)$, so that the balls collide when $t = 2$ and their common position is $(2, 1)$.

(b) We have $\mathbf{x}'(2) = (4, 2)$ and $\mathbf{y}'(2) = (1, -4)$. The angle between the paths is the angle between these tangent vectors, which is

$$\text{Arc cos}\,\frac{\mathbf{x}'(2)\cdot\mathbf{y}'(2)}{|\mathbf{x}'(2)|\cdot|\mathbf{y}'(2)|} \quad =$$

$$\text{Arc cos}\,\frac{-4}{\sqrt{20}\sqrt{17}} \quad = \quad \text{Arc cos}\,\frac{-2}{\sqrt{5}\sqrt{17}}\ .$$

27. This verification is fairly straightforward:

$$\frac{d}{dt}\,(\mathbf{x}\cdot\mathbf{y}) \quad = \quad \frac{d}{dt}\,x_1y_1 \ + \ \cdots \quad = \quad \left(\frac{dx_1}{dt}\cdot y_1 \ + \ x_1\cdot\frac{dy_1}{dt}\right) \quad =$$

$$\left(\frac{dx_1}{dt}\cdot y_1 \ + \ \cdots\right) \ + \ \left(x_1\cdot\frac{dy_1}{dt} \ + \ \cdots\right) \quad = \quad (\mathbf{x}'\cdot\mathbf{y}) \ + \ (\mathbf{x}\cdot\mathbf{y}')$$

30.(a) We need to show that $|\mathbf{x}'(t)|^2 = 1$ for all t. However, the latter is equal to $\cos^2 t + \cos^2 t \sin^2 t + \sin^4 t$ and the latter simplifies to 1 by two applications of the identity $\sin^2 + \cos^2 = 1$.

(b) The exercise suggests that one calculate the velocity vector and show that its dot product with the position vector is zero. But we have $\mathbf{v}(t) = (-\sin t, -\sin^2 t + \cos^2 t, 2\sin t \cos t)$, and direct computation shows that $\mathbf{v} \cdot \mathbf{x} = 0$.

(c) If $\mathbf{x}(t)$ is a path on the unit sphere, then $|\mathbf{x}(t)|^2 = \mathbf{x}(t) \cdot \mathbf{x}(t) = 1$ for all t, and hence by Proposition 1.7 the position vector is perpendicular to its velocity vector. [*Note:* The proof of this result appears in the discussion of Kepler's Laws, but its derivation only requires Proposition 1.4, which is verified in Exercise 27.]