Some items from the lectures on Colley, Section 3.2

RESTRICTION OF COVERAGE. The material on curvature, and everything subsequent to this, will **NOT** be covered in this course. The cutoff point is after the second full paragraph on page 194, just after the discussion of arc length parametrization. — Here is a worked out example for the latter in which the integrals can be managed using the standard methods of single variable calculus.

Example. Find the arc length parametrization for the curve

$$\mathbf{x}(t) \quad = \quad \left(4(\sin t - t\cos t),\ 4(\cos t + t\sin t),\ \tfrac{3}{2}\,t^2 \right) \ .$$

Take $t = 0$ as the initial reference point, and suppose that t runs from 0 to some positive final time F.

We know that the arc length parameter is given by the formula

$$s(t) \quad = \quad \int_0^t |\mathbf{x}'(t)|\, dt$$

and what we need to do is evaluate the right hand side of this explicitly as a function of t, after which we must solve for t in terms of s.

If we compute $\mathbf{x}'(t)$ and substitute this into the arc length formula above, we obtain

$$s(t) \quad = \quad \int_0^t \sqrt{(x_1'(u)\,)^2 + (x_2'(u)\,)^2 + (x_3'(u)\,)^2}\, du \quad =$$

$$\int_0^t \sqrt{16(\cos u - \cos u + u\sin u)^2 + 16(-\sin u + \sin u + u\cos u)^2 + (3u)^2}\, du \quad =$$

$$\int_0^t \sqrt{(4u\sin u)^2 + (4u\cos u)^2 + (3u)^2}\, du \quad = \quad \int_0^t \sqrt{16u^2 + 9u^2}\, du \quad =$$

$$\int_0^t 5u\, du \quad = \quad \tfrac{5}{2}\,t^2 \ .$$

We can now solve for t in terms of s, and the result is

$$t \quad = \quad \sqrt{\frac{2s}{5}} \ .$$

Answers to selected exercises from Colley, Section 3.2

2. The velocity is given by $\mathbf{x}'(t) = (2t, 2\sqrt{2t+1})$, so the length is given by

$$\int_0^4 \sqrt{(2t)^2 + 4(2t+1)}\, dt \quad = \quad \int_0^4 \sqrt{4t^2 + 8t + 4}\, dt \quad = \quad \int_0^4 2\,|t+1|\, dt \quad =$$

$$\int_0^4 2(t+1)\, dt \quad = \quad t^2 + \tfrac{1}{2}\,|_0^4 \quad = \quad 24 \ .$$

3. The velocity is given by $\mathbf{x}'(t) = (-3\sin 3t, 3\cos 3t, 3t^{1/2})$, so the length is given by

$$\int_0^2 \sqrt{9\sin^2 3t + 9\cos^2 3t + 9t}\,dt \;=\; \int_0^2 \sqrt{1+t}\,dt \;=\; \int_1^3 \sqrt{u}\,du \;=$$

$$2u^{3/2}\,|_1^3 \;=\; 6\sqrt{3} - 2\ .$$

4. The velocity is given by $\mathbf{x}'(t) = (0, 1, 2t)$, so the length is given by

$$\int_1^3 \sqrt{1+4^2}\,dt \;=\; \left[t\sqrt{(1/4)+t^2} + (1/4)\log\left(t + (1/4) + t^2\right)\right]_1^3 \;=$$

$$\frac{3\sqrt{37} - \sqrt{5}}{2} \;+\; \frac{1}{4}\cdot\left(\log\frac{6+\sqrt{37}}{2+\sqrt{5}}\right) \;\approx\; 8.2681459\ .$$

6. The velocity is given by $\mathbf{x}'(t) = (2\cos t - 2t\sin t, 2\sin t + 2t\cos t, 4\sqrt{2}\,t)$ so the length is given by

$$\int_0^3 \sqrt{4\cos^2 t - 8t\cos t\sin t + 4t^2\sin^2 t + 4\sin^2 t + 8t\sin t\cos t + t^2\cos^2 t + 32t^2}\,dt \;=$$

$$\int_0^3 \sqrt{4+36t^2}\,dt \;=\; \left[t\sqrt{1+9t^2} + \mathrm{Inv\ sinh}\,(3t)/3\right]_0^3 \;=\; 3\sqrt{82} + \mathrm{Inv\ sinh}\,(9)/3\ .$$

7. The length is given by

$$\int_0^{2\pi} \sqrt{9a^2\cos^4 t\sin^2 t + \sin 4t\cos 2t}\,dt \;=\; \int_0^{2\pi} \sqrt{(3a\sin^2 t\cos 2t)^2}\,dt \;=$$

$$4\cdot\int_0^{2\pi} 3a\,|\sin t\cos t|\,dt\ .$$

Notice the absolute value signs and that they make a difference in the integrand, for expression inside the absolute value signs can be negative as well as positive. This means we cannot blindly apply the standard integral formulas (if there are no absolute value signs, the integral from 0 to 2π turns out to be zero, and we know this cannot happen). Since $\sin t\cos t = \frac{1}{2}\sin 2t$, we see that the absolute value function $|\sin t\cos t|$ is periodic with period $\pi/2$, and since the expression is nonnegative between 0 and $\pi/2$ it will be enough first to integrate $3a\sin t\cos t$ between 0 and $\pi/2$ and then to multiply the answer by 4. If we do this we find that the arc length is $6a$.

12.(a) Direct calculation shows that the arc length parameter is given by

$$s(t) \;=\; \int_0^t e^{au}\sqrt{2a^2+b^2}\,du \;=\; \frac{\sqrt{2a^2+b^2}}{a}\cdot(e^{at} - 1)\ .$$

(b) If we solve the preceding equation for t in terms of s we find that

$$t \;=\; \frac{1}{a}\log\left(\frac{as}{\sqrt{2a^2+b^2}} \;+\; 1\right)\ .$$