Some items from the lectures on Colley, Section 3.3

VECTOR FIELDS. The formal definition is simple: If U is an open region in $\mathbf{R}^n$ (where $n = 2$ and 3 are the usual choices here), then a vector field is a function $\mathbf{F}$ defined on U with values in $\mathbf{R}^n$ (the same dimension as before), and we usually assume that the coordinate functions have one or two continuous partial derivatives everywhere.

The motivating examples for this concept come from the physical sciences, and two types are particularly noteworthy. One is the notion of a *force field*, where the vector field represents a physical system in which the value of the vector field at a point represents the force at that point.

Examples. Suppose that U is given by deleting some finite number of points $\mathbf{p}_1, \mathbf{p}_2. \cdots$. If we assume that there are objects at these points with given positive masses, then these objects exert a combined gravitational force at an object located at each point of U, and the value can be computed using Newton's Law of Gravitation. On the other hand, if we assume there are particles with electrical charges at these points, then they exert a combined electrostatic force on a charged particle located at each point of U, and the value can be computed using Coulomb's Law.

In particular, if we have a single point mass at the origin, so that U is the set of all points in $\mathbf{R}^n$ other than the origin, then the gravitational force field on U is given by the formula

$$\mathbf{F}(\mathbf{x}) \quad = \quad -\frac{\mathbf{x}}{|\mathbf{x}|^3} \ .$$

A second physical example involving vector fields is the notion of a *velocity field*. In this case the model is a region U which we shall assume is filled with a fluid in which the liquid is constantly in motion, and the velocity at a point of U is given by the value of the vector field at that point.

Examples. Suppose that U is the plane $\mathbf{R}^2$ and we have a fluid in which everything is moving to the right at a constant speed of c units per second. Then one has the corresponding velocity field $\mathbf{F}(x, y) = (c, 0)$. On the other hand, suppose that we have a fluid in which points are moving counterclockwise in circles about the origin, and the speed is given by the distance of the point from the origin. Then we shall see that the corresponding velocity field is given by $\mathbf{F}(x, y) = (-y, x)$. Finally, suppose that we have a river, modeled by the set U of all (x, y) such that $-1 < y < 1$, and the water is flowing to the right at a speed of $1 - y^2$ (so at the banks the water is essentially not moving and the water is moving fastest in the middle). In this case we shall see that the corresponding velocity field is given by $\mathbf{F}(x, y) = (1 - y^2, 0)$.

FINDING FLOW CURVES FOR VELOCITY FIELDS. Suppose that we have an n-dimensional velocity field $\mathbf{F}(\mathbf{x}) = \big(v_1(\mathbf{x}), \ \cdots\big)$. If we are given a moving particle whose position at time a is given by $\mathbf{b}$, then the path of the particle in the fluid is given by solving the system of differential equations

$$\frac{d\mathbf{x}}{dt} \quad = \quad \mathbf{F}(\mathbf{x}) \quad = \quad \big(v_1(\mathbf{x}), \ \cdots\big)$$

or equivalently

$$\mathbf{x}_i'(t) \quad = \quad v_i(\mathbf{x}) \ , \qquad i \ = \ 1, 2, \ \cdots$$

where the initial conditions (or constants of integration) are specified by $\mathbf{x}(a) = \mathbf{b}$. In several of the exercises, the object is to verify that a given curve is a flow curve, which means that the equations above are satisfied (and one ignores questions about initial conditions).

Exercise 19. Verify that the curve $\mathbf{r}(t) = (\sin t, \cos t, e^{2t})$ is a flow curve for $\mathbf{F}(x, y, z) = (y, -x, 2z)$. — In this case we know that $x = \sin t$, $y = \cos t$ and $z = e^{2t}$. Therefore we have $x' = \cos t = y$, $y' = -\sin t = -x$ and $z' = 2e^{2t} = 2z$.

Basic results on differential equations imply that systems like the preceding ones always have unique local solutions provided, say, the coordinate functions have continuous partial derivatives, and in some cases one can solve the systems of differential equations fairly easily using material from single variable calculus.

For example, consider the velocity field $\mathbf{F}(x, y) = (x, -y)$, and suppose we want to find the unique integral curve which passes through $(2, 3)$ when $t = 0$. In this case the flow curves are solutions to the system of differential equations $x' = 2x$, $y' = 3y$. A basic result from single variable calculus states that every function satisfying a differential equation of the form $w' = kw$ is expressible as $w = C\,e^{kt}$ for some constant of integration C. Applying this to x and y, we see that $x = Ae^t$ and Be^{-t} for some constants A and B. To find A and B, we use the initial conditions $x(0) = 2$ and $y(0) = 3$. These imply that $2 = x(0) = Ae^0 = A$ and $3 = y(0) = Be^{-0} = B$, so that the curve in question has parametric equations $(2e^t, 3e^{-t})$.

In the example of the flow $\mathbf{F}(x, y) = (1 - y^2, 0)$ we obtain the system of differential equations $x' = 1 - y^2, y' = 0$ whose general solution has the form $y = C$ (constant) and $x = (1 - C^2)t + B$, where B and C are constants of integration. For any point (p, q) in the horizontal strip and a fixed time value t_0, one can find specific values of B and C from the equations $q = C$ and $p = (1 - C^2)t_0 + B$.

The preceding examples are somewhat unusual in that one can find the parametric equations explicitly. Even in some relatively simple cases, this is not possible, so homework exercises for finding flow curves are beyond the scope of this course. The preceding material is meant to illustrate some of the ways in which vector fields arise in subjects closely related to mathematics.

Answers to selected exercises from Colley, Section 3.3

18. Since $\mathbf{r}(t) = (x, y, z) = (\sin t, \cos t, 2t)$ we have $\mathbf{x}'(t) = (\cos t, -\sin t, 2) = (y, -x, 2) = \mathbf{F}\big(\mathbf{x}(t)\big)$.

21. Students are not responsible for knowing how to work this problem.

22. Ditto.

23.(a) If $f(x, y, z) = 3x - 2y + z$, then $\nabla f = \mathbf{F}$ so $\mathbf{F}$ is a gradient field.

(b) The equipotiential surfaces are given by $3x - 2y + z = C$, where C is some constant which can be any real number. These are the planes whose normal vectors are given by $(3, -2, 1)$.

24.(a) If $f(x, y, z) = x^2 + y^2 - 3z$, then $\nabla f = \mathbf{F}$ so $\mathbf{F}$ is a gradient field.

(b) The equipotential surfaces are given by $x^2 + y^2 - 3z = C$, where C is some constant which can be any real number. These level surfaces are elliptic paraboloids formed by rotating the curves $z = \frac{1}{3}x^2 + C_1$ about the z-axis.

25. Let x be a flow line of a gradient vector field $\nabla f = \mathbf{F}$. If $G(t) = f(\,\mathbf{x}(t)\,)$, by the Mean Value Theorem it is enough to check that $G'(t)$ is always nonnegative. By the Chain Rule we have $G'(t) = \mathbf{F}(\,\mathbf{x}(t)\,) \cdot \mathbf{x}'(t)$, and since $\mathbf{x}(t)$ is a flow curve for $\mathbf{F} = \nabla f$ it follows that the $G'(t)$ is given by $|\nabla F(\,\mathbf{x}(t)\,)|^2$, which is always nonnegative.

26. First of all, we have

$$\phi(x,y,0) \quad = \quad \left(\frac{x+y}{2}\, e^0 + \frac{x-y}{2}\, e^0,\ \frac{x+y}{2}\, e^0 + \frac{y-x}{2}\, e^0 \right) \quad = \quad (x,y)\ .$$

Next, we have

$$\frac{\partial}{\partial t}\,\phi(x,y,t) \quad = \quad \left(\frac{x+y}{2}\, e^t - \frac{x-y}{2}\, e^{-t},\ \frac{x+y}{2}\, e^t - \frac{y-x}{2}\, e^{-t} \right) \quad = \quad \phi(y,x,t) \quad = \quad \mathbf{F}\big(\,\phi(x,y,t)\,\big)\ .$$

30. This uses the local uniqueness property for solutions to differential equations. To simplify the discussion we shall assume that the flow $\phi(\mathbf{x},t)$ is *complete* in the sense that the function is defined for all $\mathbf{x}$ in the region of interest and all real values of t. As usual, $\mathbf{F}$ will denote the vector field under consideration.

Let $\alpha(u) = \phi(\mathbf{x},u)$, let $\beta(s) = \phi\,(\,\phi(\mathbf{x},t)\,s\,)$, and let $\gamma(s) = \alpha(t+s)$. Then $\beta(0) = \phi(\mathbf{x},t) = \gamma(0)$. Furthermore, we have

$$\beta'(s) \quad = \quad \frac{\partial}{\partial s}\,(\,\phi(\mathbf{x},t)\,s\,) \quad = \quad \mathbf{F}(\,\beta(s)\,)\ , \quad \gamma'(s) \quad = \quad \alpha'(t+s) \quad = \quad \mathbf{F}(\,\alpha(t+s)\,) \quad = \quad \mathbf{F}(\,\gamma(s)\,)$$

so that both β and γ are flow curves for the vector field and they have the same value when $s=0$. Therefore the uniqueness statement for solutions to differential equations implies that $\gamma = \beta$ at least locally, and by our initial assumption it turns out to be true globally (unfortunately, explaining the reasons for this would take us too far afield). Therefore $\beta(s) = \gamma(s)$, and if we rewrite both sides of this equation in terms of the flow ϕ we obtain the desired identity

$$\phi\,(\,\phi(\mathbf{x},t),\ s\,) \quad = \quad \phi(\mathbf{x},t+s)\ .$$