Some items from the lectures on Colley, Section 3.4

INTERPRETING ∇–OPERATIONS ON VECTOR FIELDS. Information on these topics can be found in the files `gradient.pdf` and `weblinks3.pdf`; as usual, these are in the course directory.

Answers to selected exercises from Colley, Section 3.4

1. The divergence is given by

$$\nabla \cdot \mathbf{F} \;=\; \frac{\partial}{\partial x}\, x^2 \;+\; \frac{\partial}{\partial y}\, y^2) \;=\; 2x \;+\; 2y \;.$$

2. The divergence is given by

$$\nabla \cdot \mathbf{F} \;=\; \frac{\partial}{\partial x}\, y^2 \;+\; \frac{\partial}{\partial y}\, x^2 \;=\; 0 \;+\; 0 \;=\; 0 \;.$$

3. The divergence is given by

$$\nabla \cdot \mathbf{F} \;=\; \frac{\partial}{\partial x}\,(x+y) \;+\; \frac{\partial}{\partial y}\,(y+z) \;+\; \frac{\partial}{\partial z}\,(z+x) \;=\; 1 \;+\; 1 \;+\; 1 \;=\; 3 \;.$$

4. The divergence is given by

$$\nabla \cdot \mathbf{F} \;=\; \frac{\partial}{\partial x}\left(z\cos(e^{y^2})\right) \;+\; \frac{\partial}{\partial y}\left(x\sqrt{z^2+1}\right) \;+\; \frac{\partial}{\partial z}\left(e^{2y}\sin 3x\right) \;=$$

$$0 \;+\; 0 \;+\; 0 \;=\; 0 \;.$$

6. The divergence is given by

$$\nabla \cdot \mathbf{F} \;=\; \frac{\partial}{\partial x_1}\, x_1 \;+\; \frac{\partial}{\partial x_2}\, 2x_2 \;+\; \frac{\partial}{\partial x_3}\, 3x_3 \;\cdots\;=$$

$$1 \;+\; 0 \;+\; 0 \;+\; \cdots \;=\; 1 \;.$$

7. We shall use the symbols D_x, D_y and D_z to denote partial differentiation with respect to x, y and z in order to save space. The curl of the vector field is given by

$$\nabla \times \mathbf{F} \;=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ D_x & D_y & D_z \\ x^2 & -xe^y & 2xyz \end{vmatrix} \;=$$

$$\Big(D_y(2xyz) - D_z(xe^y),\, D_z(x^2) - D_x(2xyz),\, D_x(-xe^y) - D_y(x^2)\Big) \;=\; (2xz, -2yz, -e^y) \;.$$

8*. **(= 8 modified)** The curl of the modified vector field $\mathbf{F}(x,y,z) = (z,x,y)$ is given by

$$\nabla \times \mathbf{F} \;=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ D_x & D_y & D_z \\ z & x & y \end{vmatrix} \;=$$

$$(D_y y - D_z y, D_z z - D_x y, D_x x - D_y z) \quad = \quad (1,1,1) \ .$$

9. The curl of the vector field is given by

$$\nabla \times \mathbf{F} \quad = \quad \left| \begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ D_x & D_y & D_z \\ x + yz & y + xz & z + xy \end{array} \right| \quad =$$

$$(x - x, y - y, z - z) \quad = \quad (0,0,0) \ .$$

10. The curl of the vector field is given by

$$\nabla \times \mathbf{F} \quad = \quad \left| \begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ D_x & D_y & D_z \\ (\cos yz - x) & (\cos xz - y) & (\cos xy - z) \end{array} \right| \quad =$$

$$(D_y(\cos xy - z) - D_z(\cos xz - y), D_z(\cos yz - x) - D_x(\cos xy - z), D_x(\cos xz - y) - D_y(\cos yz - x)) \quad =$$

$$\big(x(\sin xz - \sin xy), y(\sin xy - \sin yz), z(\sin yz - \sin xz)\big) \ .$$

17. In order to minimize confusion we shall use ρ to denote $|\mathbf{r}| = r$.

By construction we have $\rho = \sqrt{x_1^2 + x_2^2 + \cdots}$ and therefore we have that

$$\frac{\partial \rho^n}{\partial x_i} \quad = \quad \frac{\partial}{\partial x_i} \left(x_1^2 + x_2^2 + \cdots\right)^{n/2} \quad =$$

$$\frac{n}{2} \left(x_1^2 + x_2^2 + \cdots\right)^{(n/2)-1} \cdot 2x_i$$

and since $(n/2) - 1 = \frac{1}{2}(n-2)$ we may rewrite the right hand side as

$$n \cdot (\rho^2)^{(n-2)/2} \cdot x_i \quad = \quad n \cdot \rho^{n-2} \cdot x_i \ .$$

Since this is the i^{th} coordinate of the gradient and x_i is the i^{th} coordinate of $\mathbf{r}$, it follows that $\nabla \rho^n = n\rho^{n-2}\,\mathbf{r}$.

18. In this case we may apply the preceding exercise to obtain

$$\nabla \rho \quad = \quad \frac{1}{\rho} \nabla \rho \quad = \quad \frac{1}{\rho} \rho^{-1}\,\mathbf{r} \quad = \quad \frac{\mathbf{r}}{\rho^2} \ .$$

19. We shall use the identity

$$\nabla \cdot (f\,\mathbf{F}) \quad = \quad f\,\nabla \cdot \mathbf{F} \ + \ (\nabla f) \cdot \mathbf{F}$$

from Exercise 23 (see below) together with the conclusion of Exercise 17. Since the divergence of $\mathbf{r}$ is equal to 3, it follows that

$$\nabla \cdot \rho^n \mathbf{r} \quad = \quad \rho^n\,\nabla \cdot \mathbf{r} \ + \ (\nabla \rho) \cdot \mathbf{r} \quad = \quad \rho^n \cdot 3 \ + \ (n\rho^{n-2})\mathbf{r} \cdot \mathbf{r}$$

and since $\rho^2 = \mathbf{r} \cdot \mathbf{r}$ it follows that the right hand side is $3\rho^n + n\rho^{n-2}\rho^2 = (n+3)\rho^n$.

20. By Theorem 4.3 on page 218, we know that $\nabla \times (\nabla f) = \mathbf{0}$ for all functions f with continuous partial derivatives. Thus it is enough to show that $\rho^n \mathbf{r}$ is the gradient of some function. The easiest way to do this is to use Exercise 18; if we set $m = n + 2$ in that result we see that $(n+2)\rho^n \mathbf{r} = \nabla \rho^{n+2}$, and the latter means that

$$\rho^n \mathbf{r} \quad = \quad \nabla \frac{\rho^{n+2}}{n+2} \ .$$

Therefore we must have $\nabla \times (\rho^n \mathbf{r}) = 0$.

23. Let $\mathbf{F}_i$ be the i^{th} coordinate of $\mathbf{F}$. Then

$$\nabla \cdot \big(f \cdot \mathbf{F}\big) \quad = \quad \sum_{i=1}^{n} \frac{\partial f \cdot F_i}{\partial x_i} \quad =$$

$$\sum_{i=1}^{n} f \, \frac{\partial F_i}{\partial x_i} \ + \ \frac{\partial f}{\partial x_i} \, F_i$$

and by the definitions the latter is equal to

$$f \, \nabla \cdot \mathbf{F} \ + \ (\nabla f) \cdot \mathbf{F} \ .$$

24. Students are not responsible for knowing how to work this problem..

28.(a) If we think of ∇ formally as (D_x, D_y, D_z), then formally $\nabla \cdot \nabla$ looks like $D_x^2 + D_y^2 + D_z^2$.

(b) If D represents partial differentiation with respect to one of x, y, z, then repeated application of the Leibniz rule implies that

$$D^2(fg) \quad = \quad (D^2 f)g \ + \ 2(Df)(Dg) \ + \ f(D^2 g) \ .$$

if we sum the appropriate terms for $D = D_x, D_y, D_z$, we see that the right hand side is

$$(D_x^2 f + D_y^2 f + D_z^2 f)g \ + \ 2(D_x f D_x g + D_y f D_y g + D_z f D_z g) \ + \ f(D_x^2 g + D_y^2 g + D_z^2 g)$$

which is a rewriting of the expression appearing in the statement of the exercise.

(c) We shall use the preceding together with Exercise 23. For any two functions u and v we have

$$\nabla \cdot (u \nabla v) \quad = \quad u \, \nabla^2 v \ + \ \nabla u \cdot \nabla v$$

and therefore we have

$$\nabla \cdot (f \nabla g) \ - \ \nabla \cdot (g \nabla f) \quad = \quad f \, \nabla^2 g \ + \ \nabla f \cdot \nabla g \ - \ (g \, \nabla^2 g \ + \ \nabla g \cdot \nabla f) \quad =$$

$$f \, \nabla^2 g \ + \ \nabla f \cdot \nabla g \ - \ g \, \nabla^2 f \ - \ \nabla g \cdot \nabla f \ .$$

The second and fourth terms cancel each other, and what remains is just $f \, \nabla^2 g \ - \ g \, \nabla^2 f$.