Answers to selected exercises from Colley, Section 3.5

1. TRUE. The condition on distance amounts to saying that $\mathbf{x} \cdot \mathbf{x} = C$ (constant), and if we differentiate both sides we get $0 = (\mathbf{x} \cdot \mathbf{x})' = 2\,\mathbf{x} \cdot \mathbf{x}'$, which is equivalent to saying that $\mathbf{x} \cdot \mathbf{x}' = 0$.

2. FALSE. If $\mathbf{x}(t) = (\cos t. \sin t)$, then $|\mathbf{x}'| = 1$ but $\mathbf{x}'(t) = (-\sin t, \cos t)$ is definitely not constant.

3. TRUE. Use Exercise 3 from Section 3.6 and the fact that $|\mathbf{x}'| = 1$ for the arc length parametrization.

30. TRUE. We show this using Exercise 25 from Section 3.4. The latter states that

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) \quad = \quad (\nabla \times \mathbf{F}) \cdot \mathbf{G} \; - \; \mathbf{F} \cdot (\nabla \times \mathbf{G})$$

(CAUTION: note the negative sign in this formula!). We want to determine whether this is zero if $\mathbf{F} = \nabla f$ and $\mathbf{G} = \nabla g$ for some functions f and g with continuous second derivatives. In this case the right hand side of the displayed equation becomes

$$(\nabla \times \nabla f) \cdot \nabla g \; - \; \nabla f \cdot (\nabla \times \nabla g)$$

and since $\nabla \times (\nabla h) = \mathbf{0}$ for all functions h with the given properties it follows the displayed expression reduces to $\mathbf{0} \cdot \nabla g - \nabla f \cdot \mathbf{0} = 0$.